GEOMETRY AND MAPPINGS OF SCREWS WITH APPLICATIONS TO THE HYBRID CONTROL OF ROBOTIC MANIPULATORS

By

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The nature and invariant properties of the elliptic polarity of screws in Euclidean space is established using a new series of mappings, central to which is a quaternion representation. The mappings are used to generalize Ball´s planar representation of the two-system of screws which is shown to be a mapping on a complex plane where the elliptic polarity induces a conformal mapping. Screw theory is applied to the hybrid control of robotic manipulators where it is demonstrated that a current theory based on "orthogonal" projection yields noninvariant results. New invariant methods of hybrid control are detailed by introducing invariant kinestatic filters.

CHAPTER 1

INTRODUCTION

> Research may start from definite problems whose importance it recognizes and whose solution is sought more or less directly by all forces. But equally legitimate is the other method of research which only selects the field of its activity and, contrary to the first method, freely reconnoitres in the search for problems which are capable of solution. Different individuals will hold different views as to the relative value of these two methods. If the first method leads to greater penetration it is also easily exposed to the danger of unproductivity. To the second method we owe the acquisition of large and new fields, in which the details of many things remain to be determined and explored by the first method. A Clebsch [1871, p. 6]

The theory of screws was first established by Ball [1900] after a quarter century of development. Although most mathematicians of the late nineteenth century were acquainted with screw theory, very few actually pursued its development. Consequently, in the early twentieth century it virtually faded into obscurity. However, with a revived interest in spatial mechanisms in the 1960s and robotics in the 1970s, screw theory is being slowly resurrected in either the guise of dual vectors or Plucker coordinates. An excellent treatment of Plucker coordinates in application to mechanisms has been given by Woo and Freudenstein [1970]. Recent texts by Hunt [1978] and Bottema and Roth [1979] have investigated and applied screws in various kinematic areas.

However, the problem that was initially investigated for this dissertation, the "orthogonality" of screws, was not treatable using the current literature in the field of kinematics. It was necessary to reconsider some of the geometrical work of the nineteenth century and try to put it in a modern perspective with improvements of notation, methods, etc. The beginning of this dissertation deals with a systematic development of the geometry of points, planes and lines to provide a broad basis for understanding the subtleties of screw theory in a unified framework. As such, the author has tried to integrate some of the pertaining accomplishments of the great late nineteenth century geometers: Ball, Cayley, Clifford, Grassmann, Klein and Plucker. Whenever possible, original papers were referred to in order to understand the spirit and implications between the lines which are forever lost in most modern presentations. Original quotes have been included throughout to compensate somewhat for this lamentable circumstance.

The geometrical development rightly begins with projective geometry, which is the most general, in Chapter 2. In order to commence on an analytical basis at the outset, coordinates are introduced in a metric-free manner for points. They are then generalized using determinant principles which elegantly display dualistic properties. Projective transformations leave incident relations invariant and are developed based on a tetrahedron principle.

Metrical geometry is developed in Chapter 3 from projective geometry based on Cayley's conception of the Absolute. Elliptic geometry is detailed with emphasis on determining elliptic relations for screws that appear similar to projective ones. Here the elliptic polarity is first introduced. Elements of Euclidean geometry are then derived with what appears to be a new contribution in deducing Euclidean collineations based directly on the invariance of the Absolute. Vectors are introduced and are used to characterize screws in particular.

Chapter 4 deals with the role of the elliptic polarity in Euclidean space. It uses the previous developments to explain the motivation for and the importance of this work. Unless specifically noted all developments in this chapter are claimed to be original. A series of new mappings of screws is introduced, central to which is a quaternion representation. Ball's planar representation of the two-system of screws is generalized as a mapping on an inversive plane where the elliptic polarity induces a conformal mapping.

In Chapter 5 screw theory is applied to the hybrid control of robotic manipulators. It is demonstrated that a current theory based on "orthogonal" projection yields noninvariant results under Euclidean translations and changes in the unit of length. By way of introducing invariant kinestatic filters, new invariant methods of hybrid control are presented.

CHAPTER 2

PROJECTIVE GEOMETRY

> Throughout the ages, from the ancient Egyptians and Euclid to Poncelet and Steiner, geometry has been based on the concept of measurement, which is defined in terms of the relation of congruence. It was von Staudt (1798-1867) who first saw the possibility of constructing a logical geometry without this concept. Since his time there has been an increasing tendency to focus attention on the much simpler relation of incidence, which is expressed by such phrases as "The point A lies on the line p" or "The line p passes through the point A." H.S.M. Coxeter [1942, p. 18]

During the nineteenth century a great deal of investigation centered about geometrical relations which are independent of metrical concepts such as distance, angle, area, etc. Metric-free geometric relations form the subject of projective geometry of which the property of incidence is paramount, such as a point is incident to a plane or a line cuts a conic. For many years projective geometry was viewed as a relatively insignificant area within the domain of Euclidean geometry. The situation took a radical turn in 1859 when Cayley demonstrated that projective geometry was actually the most general and that Euclidean geometry was merely a specialization where a certain configuration was deemed fixed and used as a reference for measurement. Later, Klein demonstrated how non-Euclidean geometries could be included under Cayley's principle which is explained in Chapter 3.

Thus, there arose a contradiction. Since coordinates were typically based upon metrical considerations, how could such coordinates be logically applied to projective relations since metrical geometry is subordinate? Klein supplied an answer to this by suggesting the use of von Staudt´s projective constructions which are employed to define the algebra of points and is presented in a modified form in Section 2.1. First nonhomogeneous coordinates are introduced which are then used to define the homogeneous coordinates of points.

In Section 2.2 determinant formulations are utilized to develop homogeneous coordinates in the plane for points and lines and in space for points, planes, lines and screws following extensionsal principles attributable to Grassmann. This enables a rather elegant and complete symmetry to be displayed amongst dual elements. Plucker´s ray and axis line coordinates are detailed and their important idential relation is formulated which is subsequently used throughout the entire work.

Projective transformations of space preserve incidence relations and are detailed in Section 2.3. An important tetrahedron relationship is introduced which unifies projective transformations for points, planes and lines or screws. Polarities are developed as correlations whose properties are symmetrical with their inverses and are then employed in Chapter 3.

Section 2.1 A Projective Formulation of Coordinates

> In the first place, it is important to realize that when coordinates are used, in projective Geometry, they are not coordinates in the ordinary metrical sense, i.e. the numerical measures of certain spatial magnitudes. On the contrary, they are a set of numbers, arbitrarily but systematically assigned to different points, like the numbers of houses in a street, and serving only, from a philosophical standpoint, as convenient designations for points which the investigation wishes to distinguish. But for the brevity of the alphabet, in fact, they might, as in Euclid, be replaced by letters. Bertrand Russell [1897, pp. 118-119]

Projective geometry deals with the incidence properties of elements and figures without employing any form of measurement or metric. However, various metrical subgeometries may be derived from the more general projective geometry by requiring that certain figures be designated as an absolute reference. Thus to employ metrical concepts, such as Cartesian coordinates, in the analytical development of projective geometry is to incur a logical contradiction since metrical relations are subordinate.

In this section, it is demonstrated that coordinates may be introduced into projective geometry in a metric-free manner. Coordinates are important in the analytical development of projective geometry since they are the tools for the derivation of compact expressions for complex relations and further, they provide insight which may not be evident from a synthetic development. Coordinates also provide an essential vehicle for numerical calculations which may be used to verify and illustrate relations unambiguously. Since coordinates are routinely used in metrical geometries it is highly desirable to demonstrate how they may be first utilized projectively and then specialized for metrical application.

The introduction of coordinates in projective geometry presented here is not intended to be an exhaustive treatment but rather an illustrative one. The intent is to draw a sharp distinction between projective and metrical concepts at the outset and to provide a logical sequence of development from projective to metrical geometry. The analysis in this section is based primarily on the complete and systematic methods of Veblen and Young [1910, 1917] to whom frequent reference is made.

It was however von Staudt [1857] who first demonstrated that analytic methods may be introduced into geometry on a strictly projective basis. For this purpose he invented the algebra of throws. The method was later simplified by Hessenberg [1905] and is presented here. First, the addition and multiplication of points on a line is defined using two special projective constructions. Then this algebra of points is demonstrated to be isomorphic to the field of real numbers and is extended to include the concept of infinity in a consistent manner. A unique real number is associated with each point on the line with the exception of a single point which assumes a correspondence with infinity. The unique real number associated with each point is called the nonhomogeneous coordinate of the point on the line. The apparently exceptional role of the point associated with infinity is removed upon the introduction of homogeneous coordinates.

In order to assign coordinates to points on a line U, it is first required to select three distinct points x_0, x_1 and x_∞ which together are referred to as a scale. By the special nature of the constructions which define addition and multiplication, the points of the scale are endowed with the properties of 0, 1 and ∞. Figures 2.1.1 and 2.1.2 illustrate the constructions for addition and multiplication which can also be used respectively to define subtraction and division.

Using the operation of addition it is possible to label all the points corresponding to integers, e.g. $x_1 + x_1 = x_2$, $x_1 + x_2 = x_3$, etc. Next, using the integers with the operation of division it is possible to label all points corresponding to rational numbers. Thus it can be shown that the points corresponding to rational numbers together with the two constructions are isomorphic to the field of rational numbers which includes the properties of associativity, communitivity, distributivity and the existence of inverses.

Next, it is necessary to show that for all points on the line there is a corresponding real number and that together with the constructions they are isomorphic to the field of real numbers. This transition is made from the rational points by invoking a Dedekind cut which is detailed in Coxeter [1942], Veblen and Young [1910].

For the notation used here, the nonhomogeneous coordinates of the points x_0, x_1, x_a, . . . are given by the subscripts 0, 1, a, . . . where the point x_∞ is excluded. It is important to note that the constructions yield

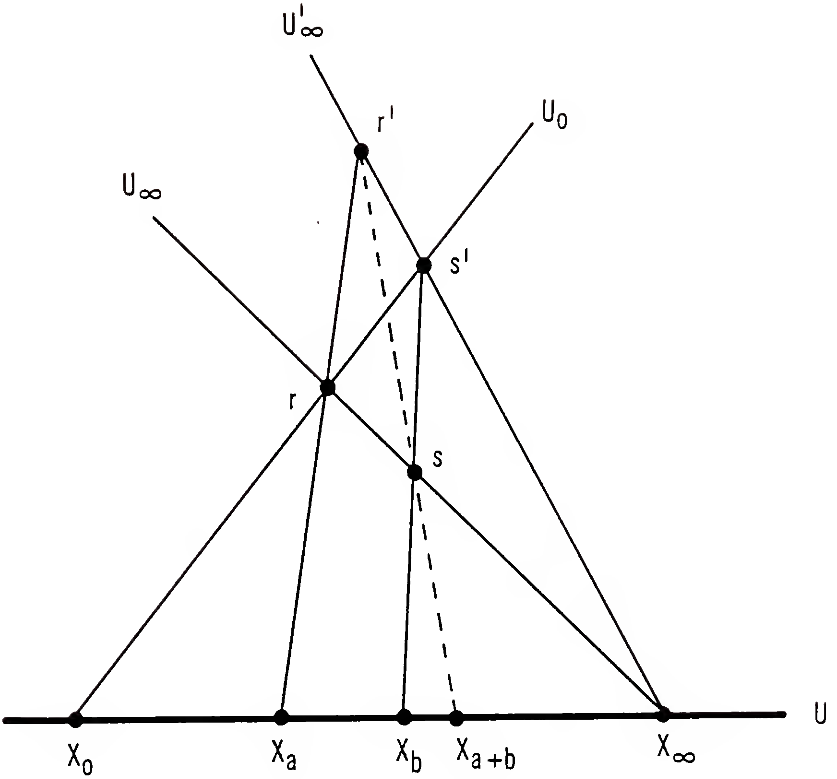

Figure 2.1.1 Addition of two points. A fixed line U_0 through x_0 meets the two distinct fixed lines U_∞ and U'_∞ in points r and s' respectively. The lines $x_a r$ and $x_b s'$ meet U'_∞ and U_∞ respectively at r' and s. The line r's meets U at x_{a+b} which yields the sum of the two points, $x_a + x_b = x_{a+b}$. By reversing the latter steps, subtraction is analogously constructed, eg. $x_a = x_{a+b} - x_b$.

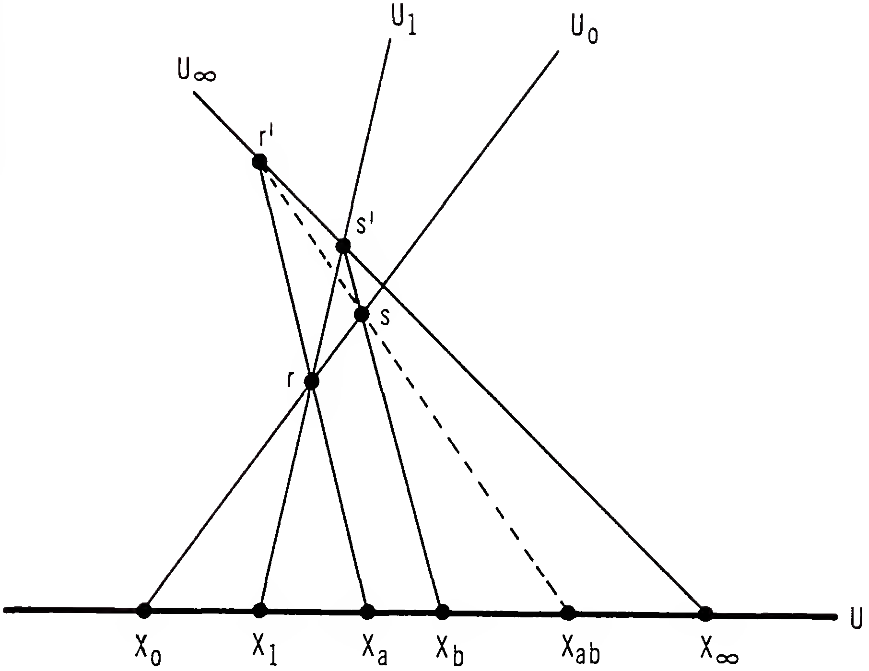

Figure 2.1.2 Multiplication of two points. Through the points x_0, x_1, x_∞ are drawn respectively the fixed lines U_0, U_1, U_∞ with U_0, U_1 meeting at r and U_1, U_∞ meeting at s'. The lines $x_a r$ and $x_b s'$ meet U_∞ and U_0 respectively at r' and s. Line r's meets U at x_{ab} which yields the product of the two points, $x_a \cdot x_b = x_{ab}$. By reversing the latter steps, division is analogously defined, eg. $x_a = x_{ab} \div x_b$.

$$x_a + x_\infty = x_\infty \tag{1}$$

$$x_a - x_\infty = x_\infty \quad (x_a \neq x_\infty) \tag{2}$$

$$x_a - x_\infty = x_\infty \quad (x_a \neq x_0) \tag{3}$$

which are consistent with the usual properties associated with infinity.

Although the three distinct points comprising the scale are selected arbitrarily, the addition and multiplication constructions impart them with the special properties associated with 0, 1 and ∞. However, from a projective standpoint, all points have identical properties. Most generally, the fundamental theorem of projective geometry, Schreier and Sperner [1935], states that in a space of dimension n, there is a unique projective transformation between a pair of n+2 elements providing that for each part of the pair no n+1 elements belong to a space of dimension n-1 (see Section 2.3 for n=3). For points on a line n=1 and thus three distinct points are related by a projective transformation. Three distinct new points may be chosen as another scale and all other points relabelled in terms of it. By way of projective transformations, all scales and subsequently all coordinates are projectively equivalent. Nevertheless, when a particular scale is employed, the scale points will still have the special properites of 0, 1 and ∞ due to the definitions of the constructions. The approach taken here is to use the addition and multiplication constructions solely for the purpose of labelling the points. These constructions are not utilized further, such as to analytically express projective transformations in terms of nonhomogeneous coordinates. Thus, the special properties of the scale points do not enter into any further analytical development.

Projective homogeneous coordinates are introduced by associating a pair of numbers a_0, a_1, which can be written as the 2×1 array $[a_0 \quad a_1]^T$, with the nonhomogenous coordinate a of the point x_a such that

$$a = \frac{a_1}{a_0} \quad . \tag{4}$$

The homogeneous coordinates of x_a are not unique since $[\lambda a_0 \quad \lambda a_1]^T$, $\lambda \neq 0$ also satisfies (4). The coordinates $[0 \quad \lambda]^T$, $\lambda \neq 0$ are associated with the point x_∞. Thus to every pair of homogeneous coordinates, with the exception of $[0 \quad 0]^T$, there corresponds a unique point of the line and to every point of the line there corresponds a pair of coordinates, which to a scalar multiple, is unique.

Projective homogeneous coordinates for points in a plane are developed from the homogeneous coordinates of points in three distinct lines. As illustrated in Fig. 2.1.3, three noncollinear points are selected as the vertices of a triangle of reference and are used to establish two scale points on each side, w_0, w_∞, x_0, x_∞, and y_0, y_∞. A fourth point which does not lie on one of the edges is selected as the unit point. Lines through the unit point and each vertex establish on the sides opposite the unit points w_1, x_1 and y_1, thus completing the three scales. Together, the three vertices and the unit

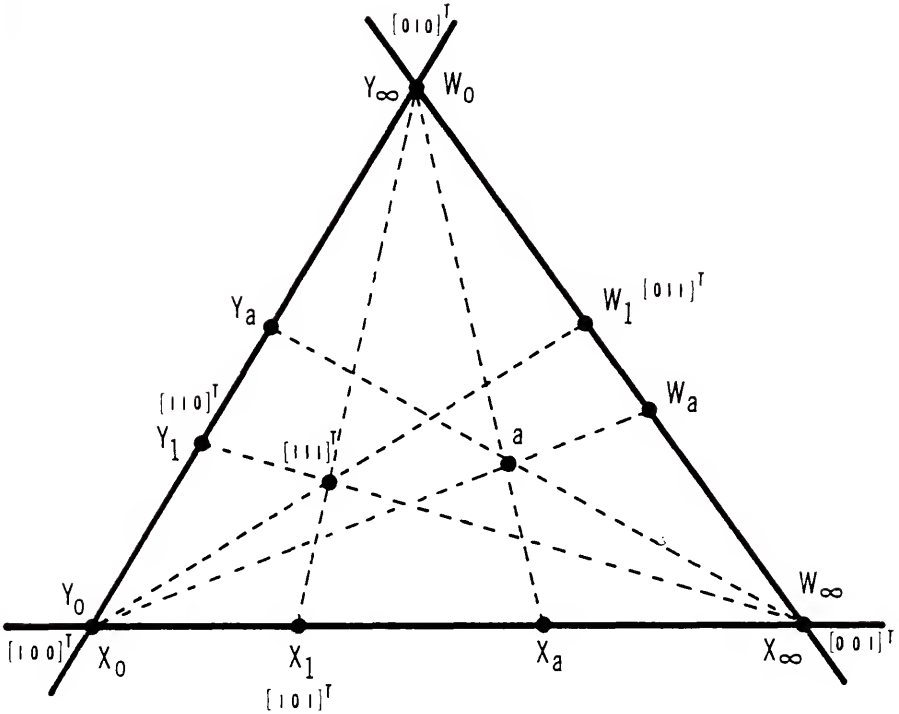

Figure 2.1.3 Projective homogeneous coordinate for the plane.

point are said to form a reference frame and respectively their homogeneous coordinates are designated by

$$[1\ 0\ 0]^T$$

$$[0\ 1\ 0]^T$$

$$[0\ 0\ 1]^T$$

and

$$[1\ 1\ 1]^T\ . \tag{5}$$

Every point on a side of the triangle may now be assigned a triple of homogeneous coordinates $[a_0\ a_1\ a_2]^T$. For example, on line U_y each of the scale points is characterized by $a_2=0$. In terms of the homogeneous coordinates of U_y, every point has the form $[a_0\ a_1]^T$ which is now denoted in the planar system by $[a_0\ a_1\ 0]^T$. Analogously, points on U_w and U_x have coordinates of the form $[0\ a_1\ a_2]^T$ and $[a_0\ 0\ a_2]^T$.

For any point not on a side of the triangle, all of its coordinates $[a_0\ a_1\ a_2]^T$ will be nonzero. Let its projections on U_x and U_y from the opposite vertices be the points x_a and y_a with coordinates determined by their respective scales, $[x_{a0}\ 0\ x_{a2}]^T$ and $[y_{a0}\ y_{a1}\ 0]^T$. It is possible to select the coordinates of the point such that

$$\frac{a_1}{a_0} = \frac{y_{a1}}{y_{a0}}\ ,\ \frac{a_2}{a_0} = \frac{x_{a2}}{x_{a0}} \tag{6}$$

that is in the ratios,

$$a_0:a_1:a_2 = x_{a0}y_{a0}:x_{a0}y_{a1}:x_{a2}y_{a0}\ . \tag{7}$$

The coordinates of x_a and y_a can thus be expressed respectively as $[a_0\ 0\ a_2]^T$ and $[a_0\ a_1\ 0]^T$. For the system of coordinates to be consistent, it is necessary to demonstrate that the coordinates of w_a, the projection of point a on U_w, are proportional to $[0\ a_1\ a_2]^T$. This can be proved using an argument involving cross ratios which is detailed in Veblen and Young [1910, pp. 175-176]. Therefore, to a nonzero scalar multiple, the projective homogeneous coordinates of a point on a plane are given by the triple $[a_0\ a_1\ a_2]^T$. It should be noted that the four points chosen for the reference frame, no three of which are collinear, are otherwise arbitrary. Thus any four such points may be chosen since by the fundamental theorem of projective geometry they are related by a unique projective transformation.

The method of using coordinates on three lines to establish coordinates on the plane leads to a recursive algorithm that may be used for a space of any dimension and is briefly described here for three dimensions.

Four noncoplanar points are chosen as the vertices of a tetrahedron of reference and a fifth point not incident with a face is selected as the unit point. As illustrated in Fig. 2.1.4, the four vertices and unit point are respectively assigned the coordinates,

$[1\ 0\ 0\ 0]^T$

$[0\ 1\ 0\ 0]^T$

$[0\ 0\ 1\ 0]^T$

$[0\ 0\ 0\ 1]^T$

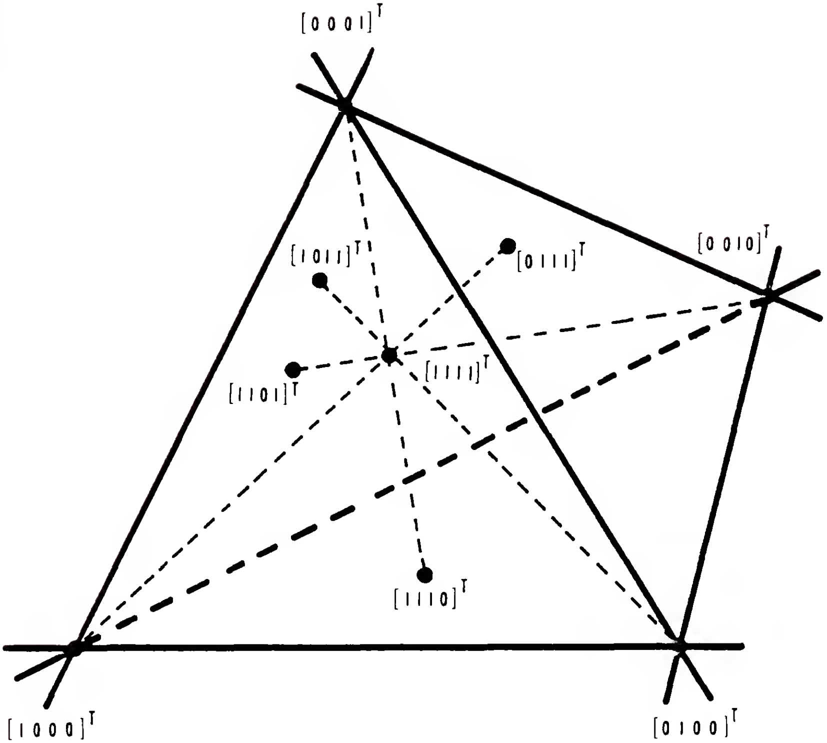

Figure 2.1.4 Tetrahedron of reference for projective homogeneous coordinates in space.

and

$$[1\ 1\ 1\ 1]^T\ . \tag{8}$$

Lines through the unit point and each vertex establish unit points on the opposite sides respectively,

$$[0\ 1\ 1\ 1]^T$$
$$[1\ 0\ 1\ 1]^T$$
$$[1\ 1\ 0\ 1]^T$$
$$[1\ 1\ 1\ 0]^T\ . \tag{9}$$

and together with the vertices, they are used to establish a planar reference system on each face.

Every point which is incident to a face of the tetrahedron may now be assigned a quadruple of homogeneous coordinates $[a_0\ a_1\ a_2\ a_3]^T$. For example, on the face opposite the vertex $[0\ 0\ 0\ 1]^T$, each of the points is characterized by $a_3=0$. This face has precisely the same planar reference system (5) described previously and thus has the form $[a_0\ a_1\ a_2]^T$ which is now denoted in the spatial system by $[a_0\ a_1\ a_2\ 0]^T$. Analogously, points on the remaining faces are characterized respectively by $a_0=0$, $a_1=0$, $a_2=0$.

For any point not incident to a face of the tetrahedron, all of its coordinates $[a_0\ a_1\ a_2\ a_3]^T$ are nonzero. It is necessary to show that the projections of the point onto the four faces can be assigned coordinates in the planar systems given respectively by

$$
\begin{aligned}
&[0\ a_1\ a_2\ a_3]^T\\
&[a_0\ 0\ a_2\ a_3]^T\\
&[a_0\ a_1\ 0\ a_3]^T\\
&[a_0\ a_1\ a_2\ 0]^T\ . \qquad (10)
\end{aligned}
$$

This has been proved by Veblen and Young [1910, pp. 194-195].

It should be noted that the five points chosen for the reference frame, no four of which are coplanar, are otherwise arbitrary. Thus any such five points may be selected since by the fundamental theorem of projective geometry they are related by a unique projective transformation.

In the preceding developments, only the projective homogeneous coordinates of points in spaces of one, two and three dimensions were considered. One-dimensional spaces are also described by lines in a plane through a point or planes through a line and an analogous procedure may be developed to assign coordinates. Further, for two-dimensional spaces such as a lines in a plane or planes through a point and for three-dimensional spaces where for example the assemblage of all planes forms a three-space, coordinates may be assigned.

In the two-dimensional space of a plane, it is possible to simultaneously assign point and line coordinates and in three-space it is possible to assign point and plane coordinates without incurring any logical inconsistencies. In the next section such a methodology is presented and is then extended by determinant principles to assign coordinates to lines and screws in projective three-space.

Section 2.2 Homogeneous Coordinates

A point given by its coordinates and a point determined by its equation, or geometrically speaking by an infinite number of planes intersecting each other in that point, are quite different ideas, not to be confounded with one another. That is the case also with regard to a plane given by its coordinates and a plane represented by its equation, or considered as containing an infinite number of points. Hence is derived a double signification of a right line. It may be considered as the geometrical locus of points, or described by a point moving along it, and accordingly represented by two equations in x, y, z each representing a plane containing that line. But it may likewise be considered as the intersection of an infinite number of planes, or as enveloped by one of these planes, turning round it like an axis; accordingly it is represented by two equations in t, u, v, each representing an arbitrary point of the line. The passage from one of the two conceptions to the other is a discontinuous one.

The geometrical constitution of space, hitherto referred either to points or to planes, may as well be referred to right lines. According to the double definition of such lines, there occurs to us a double construction of space.

In the first construction we imagine infinite space to be transversed by lines themselves consisting of points. An infinite number of such lines pass in all directions through any given point; each of these lines may be regarded as described by a moving point. This constitution of space is admitted when, in optics, we consider luminous points as sending out in all directions luminous rays, or, in mechanics, forces acting on points in every direction.

In the second construction infinite space is likewise regarded as transversed by right lines, but these lines are determined by means of planes passing through them. Every plane contains an infinite number of right lines having within it every position and direction, around each of which the plane may turn. We refer to this second conception when, in optics, we regard, instead of rays, the corresponding fronts of waves and their consecutive intersections, or when, in mechanics, according to POINSOT'S ingenious philosophical views, we

> introduce into its fundamental principles "couples," as well entitled to occupy their place as ordinary forces. The instantaneous axes of rotation are right lines of the second description. J. Plucker [1865, pp. 725-726]

The dualistic properties of projective geometry may be elegantly expressed in an analytic manner by employing homogeneous coordinates and determinant principles. The main objective of this section is the development of projective homogeneous coordinates in space, particularly line coordinates and their extensions to screws. However, it is appropriate to commence with point and line coordinates in the plane since this provides a foundation for the application of extensional principles to three-dimensional space, Forder [1940].

A line passing through two points may be described as the join of the two points and dually, the intersection point of two lines may be described as the meet of the two lines. Essentially, the principle of duality in the plane is that incidence relations remain valid when the roles of points and lines are interchanged (along with an appropriate alteration in terminology such as replacing join with meet). For example, the previous relations may be expressed as, a line (point) is the join (meet) of two points (lines).

In the plane, line coordinates are developed by first demonstrating, as in Coxeter [1942, pp. 78-80], that the condition for a point and a line to be incident may be expressed as the linear relation

$$S_0 w_0 + S_1 w_1 + S_2 w_2 = 0 \qquad (1)$$

This relation can also be expressed more compactly by

$$S^T w = 0 \qquad (2)$$

for which

$$S = [S_0\ S_1\ S_2]^T,\ w = [w_0\ w_1\ w_2]^T\ . \qquad (3)$$

The 3×1 arrays S and w are respectively projective homogeneous line and point coordinates in the plane and are unique to a nonzero scalar factor, i.e. only the ratios of the coordinates are significant. Assuming in (2) that the coordinates S have constant values while the coordinates w are free to vary, then (2) represents the locus of points which are incident to the line S, or in other words, the equation of line S. Dually, if the coordinates w are assumed constant while the coordinates S are free to vary, then (2) represents the pencil of lines which are incident to the point w, or in other words, the equation of point w.

Equation (2) may be applied to express line coordinates in terms of point coordinates and reciprocally. Let x and y be two given distinct points on S and let w be any variable point on the line. Then using (2) yields three equations

$$[w\ x\ y]^T S = 0 \qquad (4)$$

for which the vanishing of the determinant

$$|w\ x\ y| = 0 \qquad (5)$$

yields an equation in w

$$w_0|x_1\ y_2| + w_1|x_2\ y_0| + w_2|x_0\ y_1| = 0 \qquad (6)$$

where

$$|x_i\ y_j| = x_i\ y_j - x_j\ y_i \qquad (7)$$

is the ij minor in (5). Based on the expansion (6), it is useful to define the determinants of the nonsquare arrays [w] (or w) and [xy] as the 3×1 column arrays

$$|w| = [w_0\ w_1\ w_2]^T \qquad (8)$$

$$|xy| = [|x_1\ y_2|\ |x_2\ y_0|\ |x_0\ y_1|]^T. \qquad (9)$$

Thus equations (5) or (6) may be expressed simply as

$$|w|^T\ |xy| = 0 \quad \text{or} \quad w^T|xy| = 0\ . \qquad (10)$$

Comparing (10) with (2) yields the coordinates of line S to a scalar multiple in terms of points x and y,

$$S = |xy|. \qquad (11)$$

Since x and y are assumed distinct, then from (5) it is readily deduced that any point on the line S may be expressed in the form

$$w = \alpha x + \beta y \qquad (12)$$

where scalars α and β are both not simultaneously zero. Equation (12) is referred to as the freedom equation of the line and any point on it is given by the ratio $\alpha:\beta$. Using

(12) it may be shown that the coordinates of S are, to a scalar factor, independent of which two distinct points are chosen. For example, replacing x and y with $(\alpha_0 x + \beta_0 y)$ and $(\alpha_1 x + \beta_1 y)$ in (11) yields

$$S = |\alpha\beta| \cdot |xy| \tag{13}$$

where the scalar $|\alpha\beta| \neq 0$ since the points are assumed distinct.

Because of the duality principle, the development of line coordinates in terms of point coordinates is completely analogous to the preceding analysis. Let T and U be two given distinct lines incident to w, then if the variable line S is also concurrent,

$$[S\ T\ U]^T w = 0 \tag{14}$$

for which

$$|S\ T\ U| = 0 \tag{15}$$

yields an equation in S,

$$S_0|T_1\ U_2| + S_1|T_2\ U_0| + S_2|T_0\ U_1| = 0 \tag{16}$$

or equivalently,

$$|S|^T\ |T\ U| = 0\ . \tag{17}$$

Comparing (17) with (12) yields the coordinates of point w to a scalar multiple in terms of lines T and U,

$$w = |T\ U|. \tag{18}$$

Analogous to (12), the freedom equation for any line incident to w is deduced from (15)

$$S = \alpha T + \beta U \tag{19}$$

where α and β are both not simultaneously zero.

In Section 2.1 the vertices of the reference triangle were assigned point coordinates corresponding to the columns of I_3 (where I_n is the n×n identity matrix) and the first column represented what is usually called the origin. By substituting pairs of vertices in (11) it is shown that the line coordinates of any side and the point coordinates of the opposite vertex are given by the same column of I_3. The side opposite the origin is often called the line at infinity though no reference to distance is implied here.

For the plane, point and line coordinates can be summarized using

$$|x| \quad , \quad |T| \tag{20}$$

$$|x\ y| \quad , \quad |T\ U| . \tag{21}$$

In (20), point and line coordinates are given respectively by the three 1×1 determinants of [x] and [T] selected in the order 0, 1, 2. In (21) line and point coordinates are given respectively by the three 2×2 determinants of [xy] and [TU] selected in the order 12, 20, 01.

In higher order spaces many of the developments for the plane may be further generalized by application of extensional determinant principles which were first introduced by Grassmann, see Forder [1940], Klein [1908]. For three-dimensional projective space, the point may be chosen as the fundamental element and then a line is the join of two points and a plane is the join of three points. Dually, the plane may be selected as the fundamental element and then a line is the meet of two planes and the point is the meet of three planes. In space, the point and plane are dual elements whereas the line is self-dual. The duality principle in space asserts that incidence relations that are valid for points, lines and planes remain valid when their roles are interchanged with planes, lines and points along with an appropriate alteration in terminology.

The condition for a point and plane to be incident may be expressed as the linear relation

$$S_0w_0 + S_1w_1 + S_2w_2 + S_3w_3 = 0 \tag{22}$$

or equivalently

$$S^Tw = 0 \tag{23}$$

for which now

$$S = [S_0S_1S_2S_3]^T \ , \ w = [w_0w_1w_2w_3]^T \ . \tag{24}$$

The 4×4 arrays S and w are respectively projective homogeneous plane and point coordinates and are unique to a nonzero scalar factor since only the ratios of the coordinates are significant. Equation (23) may represent either the locus of points incident to plane S or dually, the bundle of planes incident to point w, the difference being respectively whether S or w is assumed fixed while the other is free to vary.

Plane coordinates may be developed from point coordinates by requiring that one variable and three given noncollinear points be incident to a plane

$$[w\ x\ y\ z]^T S = 0 \tag{25}$$

for which

$$|w\ x\ y\ z| = 0 \tag{26}$$

yields an equation in the variable point w

$$w_0|x_1y_2z_3| + w_1|x_2y_0z_3| + w_2|x_0y_1z_3| + w_3|x_1y_0z_2| = 0 \tag{27}$$

and $|x_iy_jz_k|$ is the ijk minor in (26). Alternately, (27) may be expressed more compactly using nonsquare determinants

$$|w|^T\ |xyz| = 0 \text{ or } w^T|xyz| = 0 \tag{28}$$

where

$$|w| = [w_0\ w_1\ w_2\ w_3]^T \tag{29}$$

$$|xyz| = [|x_1y_2z_3|\,|x_2y_0z_3|\,|x_0y_1z_3|\,|x_1y_0z_2|]^T \tag{30}$$

Comparing (30) with (23) yields the coordinates of plane S to a scalar factor in terms of the points x, y and z,

$$S = |x\ y\ z|. \tag{31}$$

Since x, y and z are noncollinear, then from (26) it is deduced that any point on the plane S may be expressed in the form,

$$w = \alpha x + \beta y + \gamma z \tag{32}$$

where the scalars α, β and γ are all not simultaneously zero. Equation (32) is the freedom equation of the plane and any point on it is specified by the ratios $\alpha : \beta : \gamma$. Substituting for x, y and z in (31), any other three noncollinear points on the plane S, $\alpha_i x + \beta_i y + \gamma_i z$, i=1,2,3 yields

$$S = |\alpha\beta\gamma| \cdot |xyz| \tag{33}$$

where the scalar $|\alpha\beta\gamma| \neq 0$ since the points are noncollinear.

Briefly, by the principle of duality, the development of plane coordinates from point coordinates is entirely analogous to the preceding development. Let one variable plane and three given planes, which themselves do not meet in a line, all be concurrent at a point

$$[S\ T\ U\ V]w = 0 \tag{34}$$

for which

$$|S\ T\ U\ V| = 0 \tag{35}$$

yields an equation in the variable plane S,

$$S_0|T_1U_2V_3| + S_1|T_2U_0V_3| + S_2|T_0U_1V_3| + S_3|T_1U_0V_2| = 0 \tag{36}$$

or equivalently

$$|S|^T\ |T\ U\ V| = 0. \tag{37}$$

Comparing (37) with (26) yields the coordinates of point w to a scalar factor in terms of planes T, U and V,

$$w = |T\ U\ V| . \tag{38}$$

Further, the freedom equation of the planes through w is deduced from (35),

$$S = \alpha T + \beta U + \gamma V \tag{39}$$

for α, β and γ not all simultaneously zero.

In Section 2.1 the vertices of the reference tetrahedron were assigned coordinates corresponding to the columns of I_4. By substituting triples of vertices in (31) it may be shown that the plane coordinates of any face and the point coordinates of the opposite vertex are given by the same column of I_4. For instance, the first column corresponds to the vertex at the origin and to the face that is often referred to as the plane at infinity.

In space, point and plane coordinates may be summarized and line coordinates introduced using the nonsquare determinants,

$$|x| \quad , \quad |T| \tag{40}$$

$$|x\ y| \quad , \quad |T\ U| \tag{41}$$

$$|x\ y\ z| \quad , \quad |T\ U\ V| . \tag{42}$$

In (40) point and plane coordinates are given respectively by the four 1×1 determinants of [x] and [T] selected in the order 0, 1, 2, 3. For (42), plane and point coordinates are given respectively by the four 3×3 determinants of [x y z] and [T U V] selected in the order 123, 203, 013, 102.

Plucker's ray and axis line coordinates, Plucker [1865, 1866], are defined by (41) respectively as the six 2×2 determinants of [x y] and [T U] selected in the order 01, 02, 03, 23, 31, 12. Ray line coordinates p represent the join of two points x, y and axis coordinates P represent the meet of two planes and

$$p = [p_{01}p_{02}p_{03}p_{23}p_{31}p_{12}]^T \ , \ p_{ij} = |x_i y_j| \tag{43}$$

$$P = [P_{01}P_{02}P_{03}P_{23}P_{31}P_{12}]^T \ , \ P_{ij} = |T_i U_j| \tag{44}$$

or more briefly,

$$p = |x\ y| \quad , \ P = |T\ U| \quad . \tag{45}$$

Referring to Fig. 2.2.1, the relationship between ray and axis coordinates is derived from the incidence relations of two points with two planes,

$$T^T x = 0 \tag{46}$$

$$T^T y = 0 \tag{47}$$

$$U^T x = 0 \tag{48}$$

$$U^T y = 0 . \tag{49}$$

Forming in turn, T ·(48) - U ·(46), T ·(49) - U ·(47), x ·(47) - y ·(46) and x ·(49) - y ·(48) yields respectively,

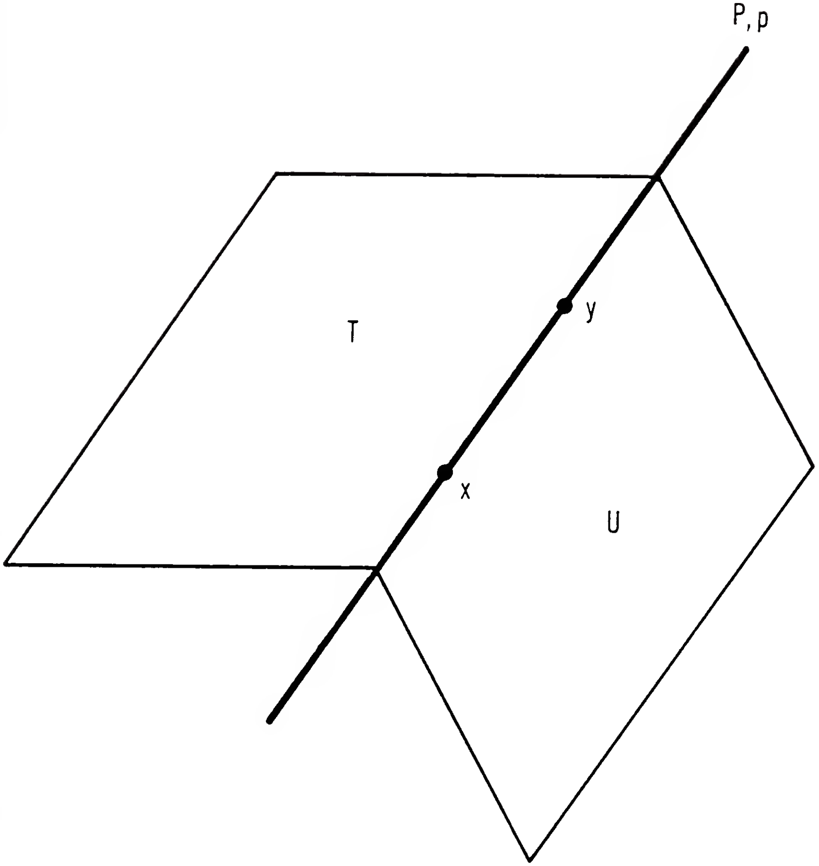

Figure 2.2.1 A line is formed as either the join of two points or the meet of two planes.

$$[P^*]x = 0 \tag{50}$$

$$[P^*]y = 0 \tag{51}$$

$$[p^*]T = 0 \tag{52}$$

$$[p^*]U = 0 \tag{53}$$

where the rank two skew-symmetric arrays are given by

$$[P^*]=\begin{bmatrix} \cdot & P_{01} & P_{02} & P_{03} \\ P_{10} & \cdot & P_{12} & P_{13} \\ P_{20} & P_{21} & \cdot & P_{23} \\ P_{30} & P_{31} & P_{32} & \cdot \end{bmatrix} \tag{54}$$

$$[p^*]=\begin{bmatrix} \cdot & p_{01} & p_{02} & p_{03} \\ p_{10} & \cdot & p_{12} & p_{13} \\ p_{20} & p_{21} & \cdot & p_{23} \\ p_{30} & p_{31} & p_{32} & \cdot \end{bmatrix} \tag{55}$$

From (50), (51) and (54), the rows (or columns) of $[P^*]$ are four planes through the line, each incident with a vertex of the reference tetrahedron. From (52), (53) and (55), the rows (or columns) of $[p^*]$ are four points on the line, each incident with a face of the reference tetrahedron.

Forming either $(50)\cdot y^T - (51)\cdot x^T$ or $(52)\cdot U^T - (53)\cdot T^T$ yields the matrix equation

$$[P^*][p^*] = 0 \ . \tag{56}$$

Table 2.2.1 Equations (57) and (58)

$$\left[\begin{array}{c:c:c:c} P_{01}P_{10}+P_{02}P_{20}+P_{03}P_{30} & P_{02}P_{21}+P_{03}P_{31} & P_{01}P_{12}+P_{03}P_{32} & P_{01}P_{13}+P_{02}P_{23} \\ P_{12}P_{20}+P_{13}P_{30} & P_{10}P_{01}+P_{12}P_{21}+P_{13}P_{31} & P_{10}P_{02}+P_{13}P_{32} & P_{10}P_{03}+P_{12}P_{23} \\ P_{21}P_{10}+P_{23}P_{30} & P_{20}P_{01}+P_{23}P_{31} & P_{20}P_{02}+P_{21}P_{12}+P_{23}P_{32} & P_{20}P_{03}+P_{21}P_{13} \\ P_{31}P_{10}+P_{32}P_{20} & P_{30}P_{01}+P_{32}P_{21} & P_{30}P_{02}+P_{31}P_{12} & P_{30}P_{03}+P_{31}P_{13}+P_{32}P_{23} \end{array}\right] = [0] \quad (57)$$

$$\begin{array}{cccc} \cdot & \frac{P_{02}}{P_{31}} = \frac{P_{03}}{P_{12}} & \frac{P_{03}}{P_{12}} = \frac{P_{01}}{P_{23}} & \frac{P_{01}}{P_{23}} = \frac{P_{02}}{P_{31}} \\ \frac{P_{31}}{P_{02}} = \frac{P_{12}}{P_{03}} & \cdot & \frac{P_{31}}{P_{02}} = \frac{P_{01}}{P_{23}} & \frac{P_{01}}{P_{23}} = \frac{P_{12}}{P_{03}} \\ \frac{P_{12}}{P_{03}} = \frac{P_{23}}{P_{01}} & \frac{P_{23}}{P_{01}} = \frac{P_{02}}{P_{31}} & \cdot & \frac{P_{12}}{P_{03}} = \frac{P_{02}}{P_{31}} \\ \frac{P_{31}}{P_{02}} = \frac{P_{23}}{P_{01}} & \frac{P_{23}}{P_{01}} = \frac{P_{03}}{P_{12}} & \frac{P_{03}}{P_{12}} = \frac{P_{31}}{P_{02}} & \cdot \end{array} \quad (58)$$

In Table 2.2.1, (56) has been expanded into components to yield (57). Then the off-diagonal terms are used to form the twelve equations in (58) written as a 4×4 array. Comparing in turn five equations in (58) given by the positions 30, 10, 13, 03, 01 yields

$$\frac{P_{23}}{P_{01}} = \frac{P_{31}}{P_{02}} = \frac{P_{12}}{P_{03}} = \frac{P_{01}}{P_{23}} = \frac{P_{02}}{P_{31}} = \frac{P_{03}}{P_{12}} = \mu \qquad (59)$$

where μ is a nonzero scalar.

It should be noted that it is not necessary to explicitly include μ since it is included implicitly by only ascribing significance to the ratios of homogeneous coordinates. Thus setting $\mu=1$ and arranging (59) in matrix form yields

$$P = \tilde{\Delta} p \qquad (60)$$

$$p = \tilde{\Delta} P \qquad (61)$$

where

$$\tilde{\Delta} = \begin{matrix} \cdot & I_3 \\ I_3 & \cdot \end{matrix} \qquad (62)$$

$$\tilde{\Delta}^T = \tilde{\Delta}, \quad \tilde{\Delta}\tilde{\Delta} = \hat{I}_6. \qquad (63)$$

Since in (60) and (61), p and P are derived from dual elements, the induced linear transformation $\tilde{\Delta}$ is a correlation which is signified by tilda. The existence of this correlation is due to the fact that in three-dimensional space lines are self-dual elements and thus $\tilde{\Delta}$ represents the identical correlation of lines. The matrix $\tilde{\Delta}$ represents a very simple method of transforming between ray and axis coordinates, the first and last three components are merely exchanged. Since a double application of the exchange yields back the initial values, then as expressed by (63), $\tilde{\Delta}$ is clearly an involution matrix.

In ray coordinates, the condition for two lines to be incident may be obtained by letting p be the join of w, x and q be the join of yz. When the lines intersect then all four points are coplanar and

$$|w\ x\ y\ z| = 0 \qquad \text{repeated, (26)}$$

which is expanded by the first two columns to yield

$$p^T\ \tilde{\Delta}\ q = 0 \quad \text{or} \quad |w\ x|^T\ \tilde{\Delta}\ |y\ z| = 0. \tag{64}$$

Alternatively, in axis coordinates let P and Q be the same two lines where P is the meet of X, T and Q is the meet of U, V. When the lines intersect then all four planes are concurrent at a point and the expansion of

$$|S\ T\ U\ V| = 0 \qquad \text{repeated, (34)}$$

yields

$$P^T\ \tilde{\Delta}\ Q = 0 \quad \text{or} \quad |S\ T|^T\ \tilde{\Delta}|U\ V| = 0. \tag{65}$$

Using (60), (61) with (64), (65) gives alternative expressions for intersection,

$$p^T Q = 0 \quad \text{or} \quad |w\ x|^T\ |U\ V| = 0 \tag{66}$$

$$P^T q = 0 \quad \text{or} \quad |S\ T|^T\ |x\ y| = 0. \tag{67}$$

Because dual coordinates are used in (66), (67) their form is analogous to (23) which expresses the incidence of a point and a plane.

Since only the ratios of homogeneous coordinates are significant, the six coordinates of a line represent five parameters. However, only four independent parameters are required to specify a line in space and therefore six coordinates must be related by a single equation. In terms of ray and axis coordinates, this relation is obtained by expanding the singular determinants

$$|xy\ xy| = 0 \quad , \quad |TU\ TU| = 0 \tag{68}$$

to yield

$$p^T\ \tilde{\Delta}\ p = 0 \quad , \quad P^T\ \tilde{\Delta}\ P = 0 \tag{69}$$

or equivalently

$$p^T P = P^T p = 0. \tag{70}$$

It is interesting to examine the line coordinates corresponding to the tetrahedron of reference. Referring to Fig. 2.2.2, the vertices along with their opposing faces are both labelled 0, 1, 2, 3 and their coordinates correspond in order to the columns of I_4. Forming the ray coordinates of the six joins of vertex pairs and the six meets of face

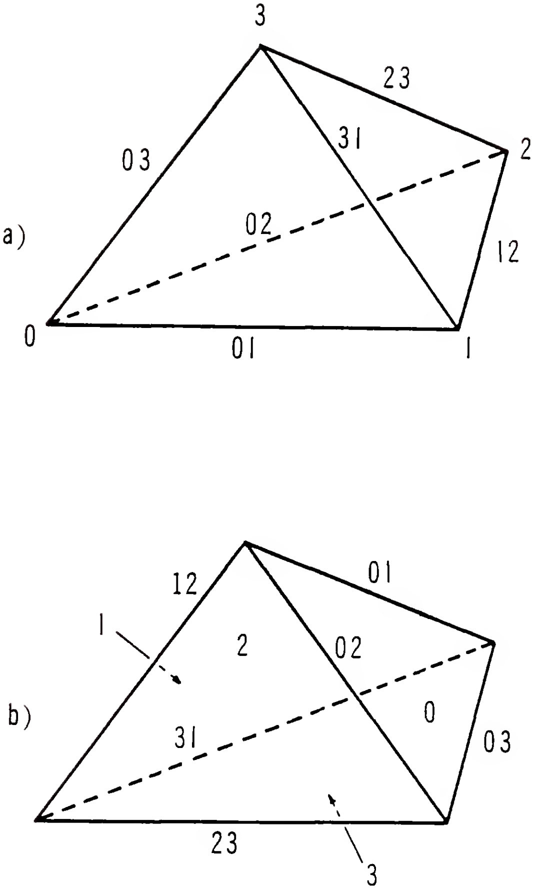

Figure 2.2.2 The tetrahedron of reference labelled with
a) point coordinates and ray coordinates,
b) plane coordinates and axis coordinates.

pairs both in the order 01, 02, 03, 23, 31, 12, then the resulting coordinates of both sets correspond in order to the columns of I_6. Opposite edges of the tetrahedron (i.e. nonintersecting), one expressed in ray coordinates and the other in axis coordinates, correspond to the same column of I_6.

Unlike point and plane coordinates, there are in general no freedom equations for lines using line coordinates corresponding to (32) and (39). (There are however freedom equations for lines in terms of either two points or two planes.) For example, let q,r be two lines and μ,λ be two scalars and consider

$$p = \mu q + \lambda r \; . \tag{71}$$

For p to be a line it must satisfy the quadratic form of (69). However,

$$p^T \tilde{\Delta} p = 2 \mu \lambda q^T \tilde{\Delta} r \tag{72}$$

which only vanishes when q and r intersect.

In general, a linear combination of lines in either ray or axis coordinates is defined here as a screw, which includes lines as special cases. Since the homogeneous coordinates of a screw need not satisfy (69), it follows that there are ∞^5 screws in space. When two screws satisfy any of the equivalent relations

$$p^T \tilde{\Delta} q = 0 \quad , \quad P^T \tilde{\Delta} Q = 0 \tag{73}$$

$$p^T Q = 0 \quad , \quad P^T q = 0 \tag{74}$$

the screws are said to be reciprocal, a property which is analogous to incidence.

For a general linear combination of n screws (or lines), $1 \leq n \leq 6$, the freedom equations

$$p = \lambda_1 q + \ldots \lambda_n r \tag{75}$$

$$P = \lambda_1 Q + \ldots \lambda_n R$$

where $\lambda_1 \ldots \lambda_n$ are all not simultaneously zero, are said to describe an n-system of screws, Ball [1900], Hunt [1978].

Screws may also be viewed abstractly as points in a five-dimensional space as suggested by F. Klein (see Jessop [1903]). Lines in this space lie on a surface which is given by (69). Dually, hyperplanes in this space correspond to five-systems of screws, which also may be described by six homogeneous coordinates. By generalizing the development of coordinates presented here, systems of coordinates may be systematically derived for n-systems, $1 \leq n \leq 5$. Detailed references on the generation of extensional systems of coordinates based on determinant properties are given by Forder [1940], Sommerville [1929], Hodge and Pedoe [1947, 1952], but will not be considered here further.

Section 2.3 Projective Transformations

> . . . I shall enunciate two general principles which I have habitually emphasized and have put into the foreground in these fundamental geometric discussions. Although in this generality they sound at first somewhat obscure, they will, with concrete illustrations, soon become clear. One of them is that the geometric properties of any figures must be expressible in formulas which are not changed when one changes the coordinate system, i.e. when one subjects all the points of the figure simultaneously to one of our transformations; and, conversely, any formula which, in this sense, is invariant under the group of these coordinate transformations must represent a geometric property. As simplest examples, which all of you know, let me remind you of the expression for the distance or for the angle, in the figure of two points or of two lines. We shall have to do repeatedly with these and with many other similar formulas in the following pages. For the sake of clearness, I shall give a trivial example of non-invariant formulas: The equation $y = 0$, for the figure consisting of the point (x,y) of the plane, says that this point lies on the x axis, which is, after all, a thoroughly unessential fact, foreign to the nature of the figure, useful only in serving to describe it. Likewise, every non-invariant equation represents some relation of the figure to external, arbitrarily added, things, in particular to the coordinate system, but it does not represent any geometric property of the figure.
>
> The second principle has to do with a system of analytic magnitudes which are formed from the coordinates of points 1, 2, . . . , such as X, Y, and N, for example. If this system has the property of transforming into itself, in a definite way, under a transformation of coordinates, i.e., if the system of magnitudes formed from the new coordinates of the points 1, 2, . . . , expresses itself in terms exclusively of these magnitudes formed in the same way from the old coordinates (the coordinates themselves not appearing explicitly), then we say that the system defines a new geometric configuration, i.e., one which is independent of the coordinate system. In fact, we shall classify all analytic expressions according to their behavior under coordinate transformation, and we shall define as geometrically equivalent two series of expressions which transform in the same way. Felix Klein [1908, pp. 25-26]

This section examines linear projective transformations of homogeneous coordinates. Klein [1872] has enunciated a definition of geometry which, except for minor extensions, is still very applicable today. Essentially, Klein stated that a geometry is defined as the properties of a space which remain invariant under all transformations of space (or the coordinate system) by a group of transformations.

For projective geometry, the group of transformations is characterized by those which preserve relations of incidence. Commencing with the group of projective point collineations, the corresponding induced collineations for planes, lines and screws are developed with respect to an elegant tetrahedronal principle employing determinant relations. Using a simple device, many of the results for collineations are extended to the nongroup of correlations. An analysis of projective transformations not only identifies important invariant relations but also forms a foundation for developing metrical geometries in Chapter 3.

A collineation is a one-to-one linear transformation in which each element of space is mapped into a corresponding element of the same type (e.g. point to point) whereas a correlation differs in that each element is mapped into a corresponding dual element (e.g. point to plane). A projective transformation is uniquely determined by five pairs of corresponding points in space provided that no four of the five points in either pair are coplanar. For the collineation,

$$\mu t = Kx \tag{1}$$

where μ is included explicitly as a factor of proportionality and where the 4×4 matrix K is given by

$$K = [A\ B\ C\ D]^T . \tag{2}$$

Since only the ratios of homogeneous coordinates are significant, the four equations in (1) can be reduced to three ratios of equations by, for example, dividing the last three equations by the first equation and thus the explicit factor μ is eliminated. Multiplying out the ratios and expressing them in matrix form yields

$$\begin{bmatrix} -t_1 x^T & t_0 x^T & . & . \\ -t_2 x^T & . & t_0 x^T & . \\ -t_3 x^T & . & . & t_0 x^T \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = [0] \tag{3}$$

where $t_0 \ . \ . \ . \ t_3$ are the coordinates of t and where the 3×16 matrix multiplies the 16×1 column array containing the unknown coefficients. Substitution for t and x by five pairs of corresponding points yields 15 homogeneous equations which are sufficient to solve for 15 ratios involving the elements of K. Thus the projective collineation is uniquely determined to a scalar factor and K is nonsingular since the mapping is one-to-one.

In (1) the factor μ was explicitly included to facilitate in the solution for K. However, it is convenient to absorb the factor by substituting $\mu=1$ which is permissible provided it is understood that only the ratios are significant,

$$t = Kx. \tag{4}$$

A projective collineation of points also induces a projective collineation of planes which may be determined using incidence properties. Let x be incident to plane T,

$$T^T x = 0\ . \tag{5}$$

The induced transformation k maps T into another plane X

$$X = kT \tag{6}$$

such that incidence is preserved,

$$X^T t = 0. \tag{7}$$

Substituting (4), (6) in (7),

$$T^T (k^T K) x = 0 \tag{8}$$

and comparing with (5) yields

$$k = K^{-T} \tag{9}$$

to an arbitrary nonzero scalar multiple. Matrix k can be calculated by replacing each element of K with its cofactor (signed minor), and dividing by the scalar $|K| = |A\ B\ C\ D|$ (although this last step is not essential),

$$k = [|B\ C\ D|\ |C\ A\ D|\ |A\ B\ D|\ |B\ A\ C|]^T / |A\ B\ C\ D| \tag{10}$$

or more simply as

$$k = [a\ b\ c\ d]^T . \tag{11}$$

The four 4×3 determinants in (10) have been formed from ABCD in the order 123, 203, 013, 102 which is also the same order used in expanding each of these nonsquare determinants into components, see (2.2.42).

There is a useful geometric interpretation for K and k. Let ABCD represent the coordinates of four planes whose equations can be written as

$$[A\ B\ C\ D]^T w = 0. \tag{12}$$

Since K is nonsingular then (12) has no solution other than w=0 which does not represent a point in homogeneous coordinates. Thus the four planes do not have a common point and they therefore form a tetrahedron. In (10), each row is the meet of three planes and is thus a vertex of the same tetrahedron. The vertices abcd are respectively opposite the faces ABCD since (2), (9) and (11) yield the incidence relations

$$[A\ B\ C\ D]^T\ [a\ b\ c\ d] = I_4 . \tag{13}$$

Additionally, K can be expressed in terms of k and by analogy with (10),

$$K = [|b\ c\ d|\ \ |c\ a\ d|\ \ |a\ b\ d|\ \ |b\ a\ c|]^T/|a\ b\ c\ d| . \tag{14}$$

The collineation of points not only induces a collineation of planes, but also induces a collineation of lines. Let x,y and t,u be a pair of corresponding points

$$t=Kx \ , \ u=Ky \ . \tag{15}$$

Forming the join of each pair, then the lines are expressed in ray coordinates by

$$p = |xy| \ , \ q = |tu| \tag{16}$$

and the line p is transformed into the line q. Substituting (15) into (16) yields

$$q = |Kx \ Ky| \tag{17}$$

and then substituting (2) in (17) gives the nonsquare determinant

$$q = \begin{vmatrix} A^Tx & A^Ty \\ B^Tx & B^Ty \\ C^Tx & C^Ty \\ D^Tx & D^Ty \end{vmatrix} \ . \tag{18}$$

The first coordinate of q is given by

$$\begin{aligned} q_{01} &= A^TxB^Ty - A^TyB^Tx \\ &= A^T[xy^T - yx^T]B \\ &= A^T[p^*]B \end{aligned} \tag{19}$$

where $[p^*]$ is the skew-symmetric metrix given in (2.2.55) where elements are $p_{ij} = |x_iy_j|$. Expanding the bilinear expression in (19) yields after some manipulations,

$$q_{01} = |AB|^T p. \tag{20}$$

The remaining components of q are determined by analogy with (20) which yields

$$q = [|AB|\ \ |AC|\ \ |AD|\ \ |CD|\ \ |DB|\ \ |BC|]^T p \tag{21}$$

and which is more concisely expressed by

$$q = \hat{K}p. \tag{22}$$

The six 4×2 determinants in (21) have been formed from the planes ABCD in the order 01, 02, 03, 23, 31, 12 which is also the same order used in expanding each of the nonsquare determinants into components, see (2.2.41).

Analogously, the induced collineation for axis coordinates can be developed from a pair of corresponding planes,

$$X = kT\ ,\ Y = kU\ . \tag{23}$$

Forming the meet of each pair in axis coordinates,

$$L = |TU| \qquad M = |XY| \tag{24}$$

then the collineation transforms line L into line M and using (23) in (24) yields

$$M = |kT\ kU|. \tag{25}$$

Expanding the terms in (25) and using (11) gives a result which is analogous to (21),

$$M = [|ab| \ |ac| \ |ad| \ |cd| \ |db| \ |bc|]^T L \tag{25}$$

or more concisely,

$$M = \hat{k} \ L. \tag{26}$$

Since a collineation preserves incidence properties, then if p and L are incident then so are q and M,

$$L^T p = 0 \tag{27}$$

$$M^T q = 0 \ . \tag{28}$$

Substitution of (22), (26) in (28) yields

$$L^T (\hat{k}^T \hat{K}) p = 0 \tag{29}$$

which leads to a result analogous to (9),

$$\hat{k} = \hat{K}^{-T} \tag{30}$$

upon comparing (27) and (29) for general intersecting lines L,p. Although (30) is correct to a scalar multiple, it can be shown that it is actually an algebraic identity when the plane collineation is given by (2) and (14) and the point collineation is given by (10) and (11).

Figure 2.3.1 is used to illustrate the geometric interpretations of the induced ray and axis line collineations. From (21), the collineation of ray coordinates is given by $\hat{K}$ whose rows are the axis coordinates for the six edges of the illustrated tetrahedron. From (25), the collineation of

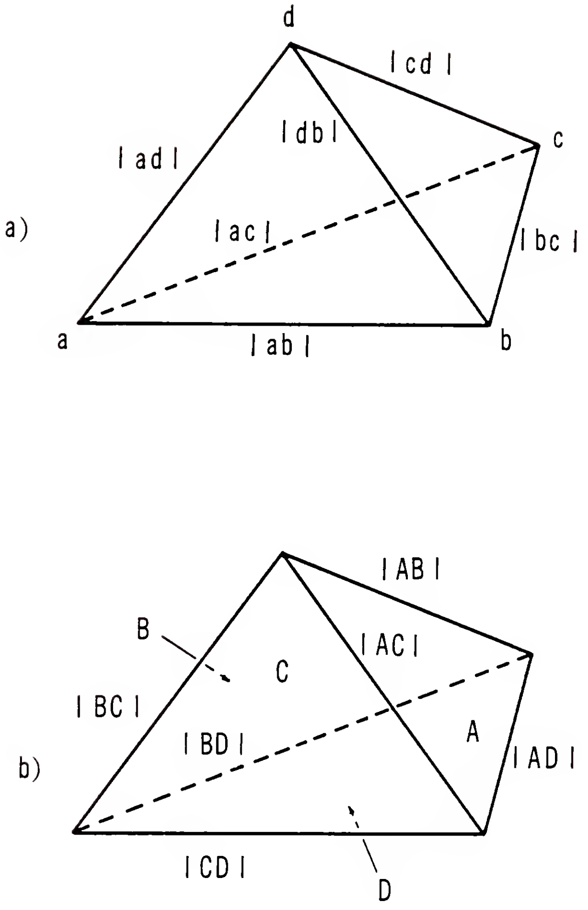

Figure 2.3.1 The tetrahedron associated with a collineation labelled with a) point coordinates and ray coordinates, b) plane coordinates and axis coordinates.

axis coordinates is given by $\hat{k}$ whose rows are the ray coordinates for the six edges of the tetrahedron. Thus the same tetrahedron is intimately related to the four collineations K, k, $\hat{K}$, $\hat{k}$.

This tetrahedron relationship leads to two important algebraic identities for the induced line collineations,

$$\hat{K}^T \tilde{\Delta} \hat{K} = |A\ B\ C\ D| \tilde{\Delta} \tag{31}$$

$$\hat{k}^T \tilde{\Delta} \hat{k} = |a\ b\ c\ d| \tilde{\Delta}. \tag{32}$$

where

$$\tilde{\Delta} = \begin{bmatrix} \cdot & I_3 \\ I_3 & \cdot \end{bmatrix} . \tag{2.2.62}$$

Briefly, the identities (31), (32) are the conditions for the rows of $\hat{K}$ and $\hat{k}$ to be edges of the tetrahedron in Fig. 2.3.1. These identities are demonstrated by first substituting (30) into (31), (32) to yield

$$|a\ b\ c\ d| \tilde{\Delta} = \hat{k} \tilde{\Delta} \hat{k}^T \tag{33}$$

$$|A\ B\ C\ D| \tilde{\Delta} = \hat{K} \tilde{\Delta} \hat{K}^T \tag{34}$$

and where additionally

$$|a\ b\ c\ d| \cdot |A\ B\ C\ D| = 1. \tag{35}$$

Relations (33), (34) are easily proved by substituting in the values of $\hat{k}$ and $\hat{K}$ in (21) and (25) and by noting that,

$$|ab|^T \tilde{\Delta}\ |cd| = |a\ b\ c\ d| \tag{36}$$

$$|AB|^T \tilde{\Delta}\ |CD| = |A\ B\ C\ D| \tag{37}$$

since the left sides are merely the Laplacian expansions of the right sides by the first two columns. Any other product such as

$$|ab|^T\ \tilde{\Delta}\ |ad| = |a\ b\ a\ d| = 0 \tag{38}$$

clearly vanishes since two columns are identical.

Using incidence properties, (31) and (32) are now verified to a scalar multiple. Since p, q, L, M are lines then they satisfy the identical relations given by (2.2.69),

$$p^T\ \tilde{\Delta}\ p = 0 \qquad\qquad L^T\ \tilde{\Delta}\ L = 0 \tag{39}$$

$$q^T\ \tilde{\Delta}\ q = 0 \qquad\qquad M^T\ \tilde{\Delta}\ M = 0. \tag{40}$$

Substituting (22), (26) into (40) yields

$$p^T(\hat{K}^T\ \tilde{\Delta}\ \hat{K})p = 0 \qquad L^T(\hat{k}^T\ \tilde{\Delta}\ \hat{k})L = 0. \tag{41}$$

Comparing (39) and (41) for independent p and L yields (31), (32) to a nonzero scalar multiple,

$$\hat{K}^T\ \tilde{\Delta}\ \hat{K} = \mu\tilde{\Delta} \qquad\qquad \hat{k}^T\ \tilde{\Delta}\ \hat{k} = \mu\tilde{\Delta}. \tag{42}$$

Equation (42) (or (31), (32)) is important since it yields the conditions which are necessary for a 6×6 matrix to represent an induced projective collineation of lines.

A 6×6 matrix contains 36 elements and since only the ratios are significant there are 35 independent parameters. In (42), either of the equations represents only 21 different scalar equations since the matrix equations are symmetric. Because these equations are not all homogeneous, the scalar multiplier μ is eliminated by considering the 20 ratios of equations (in a manner analogous to (3)). Thus the 35 parameters are related by 20 constraint equations to yield 35-20=15 independent parameters to describe an induced projective collineation of lines. This is in agreement with the projective point and plane collineations whose associated 4×4 matrices yield 15 independent ratios.

A linear combination of lines, as previously defined by

$$p = \lambda_1 q + \ldots \lambda_n r \tag{2.2.75}$$

$$P = \lambda_1 Q + \ldots \lambda_n R \tag{2.2.76}$$

is in general a screw. The induced projective collineation of screws in space is identical to that for lines by way of linearity,

$$\hat{K}p = \lambda_1 \hat{K}q + \ldots \lambda_n \hat{K}r\ , \quad \hat{k}P = \lambda_1 \hat{k}Q + \ldots \lambda_n \hat{k}R. \tag{43}$$

If a 6×6 matrix does not satisfy (42) then it is not an induced projective collineation of lines. It is possible for this to occur in two ways. Firstly, if the rows of the matrix all represent lines then they cannot form a tetrahedron. Secondly, if any row is a screw then the matrix is not an induced collineation. General transformations of this type have been investigated by Ball [1900], which he called homographic transformations. It should be noted that in the literature the terms homographic transformations and projective transformations are often used synonymously but are used distinctively here. Effectively, Ball treated screws as points in a projective five-space and therefore a homographic collineation is the most general one-to-one linear transformation of these points.

It is not difficult to verify that nonsingular projective point and plane collineations each form a group of transformations under the operation of composition or matrix multiplication. Nonsingular induced projective line collineations also form a group of transformations and it is useful to demonstrate the property of closure. Using ray coordinate transformations let $\hat{K}$, $\hat{J}$ be two nonsingular collineations and it is necessary to show that $\hat{K}\hat{J}$ is also such a collineation, i.e. it satisfies (42),

$$\begin{aligned}(\hat{K}\hat{J})^T \tilde{\Delta} (\hat{K}\hat{J}) &= \hat{J}^T (\hat{K}^T \tilde{\Delta} \hat{K}) \hat{J} \\ &= \mu \hat{J}^T \tilde{\Delta} \hat{J} \\ &= \mu\lambda\tilde{\Delta} \qquad (44)\end{aligned}$$

where $\mu\lambda$ is a nonzero scalar multiplier.

Since induced collineations constitute a group, then it follows from (42) that the bilinear forms,

$$p^T \tilde{\Delta} q \quad , \quad P^T \tilde{\Delta} Q \tag{45}$$

and in particular the quadratic forms

$$p^T \tilde{\Delta} p \quad , \quad P^T \tilde{\Delta} P \tag{46}$$

are invariant expressions with respect to induced collineations. From linear principles, this is true whether p,q (P,Q) are lines or screws. The forms (45) are often referred to as the mutual moment of two lines or screws although no metrical connotation is implied here. When screws are considered as points in a five-space then the lines of three-space are represented as points on the quadric surface

$$p^T \tilde{\Delta} p = 0 \quad , \quad P^T \tilde{\Delta} P = 0 \tag{47}$$

which is sometimes called a Grassmannian, Hodge and Pedoe [1952]. In the group of homographic transformations of screws (i.e. nonsingular 6×6 matrices), induced projective line transformations constitute a subgroup which leaves the quadric (47) invariant (transforms into itself).

Projective transformations are classified as either collineations or correlations. Correlations are linear one-to-one transformations which map each element into a dual element. Since the product of two correlations is a collineation, it follows that correlations do not possess the property of closure and thus do not form a group. However, collectively collineations and correlations form the group of projective transformations.

In the development of correlations it is useful to first define two distinct spaces where elements in one are denoted by a prime. Consider the incidence of a point x on a plane T,

$$T^T x = 0 \tag{47}$$

and the correlation,

$$X' = \tilde{\Lambda} x \quad , \quad t' = \tilde{\lambda} T. \tag{48}$$

Since projective transformations must preserve incidence relations then,

$$t'^T X' = T^T (\tilde{\lambda}^T \tilde{\Lambda}) x = 0 \tag{49}$$

and by comparing (47) and (49) then to a scalar multiple,

$$\tilde{\lambda} = \tilde{\Lambda}^{-T} . \tag{50}$$

The inverse transformations between the two spaces are given by inverting (48),

$$T = \tilde{\lambda}^{-1} t' \quad , \quad x = \tilde{\Lambda}^{-1} X' \tag{51}$$

or equivalently using (50),

$$T = \tilde{\Lambda}^T t' \quad , \quad x = \tilde{\lambda}^T X' . \tag{52}$$

Figure 2.3.2 illustrates the mapping between the two spaces described by (48) and (52). It is noted that the two point

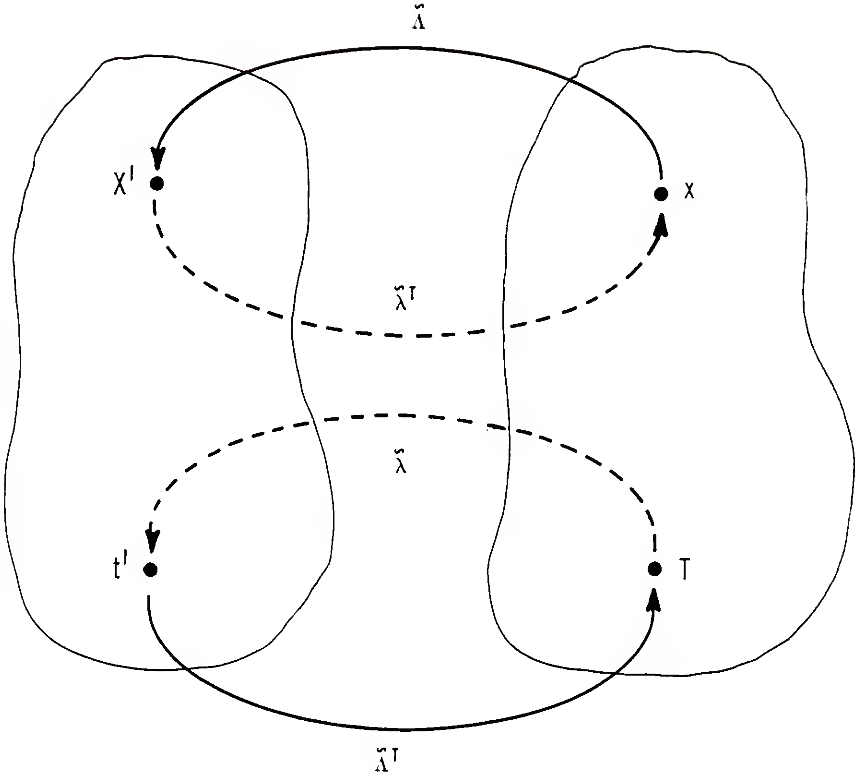

Figure 2.3.2 The transformation of two distinct spaces under a correlation.

to plane transformations (solid lines) are transposes of each other as are the two plane to point transformations (dotted lines).

Correlations of a single space onto itself may be deduced by allowing the two distinct spaces to coincide. For the correlation to be well-defined it is necessary that

$$\mu\tilde{\Lambda} = \Lambda^T \quad , \quad \mu\tilde{\lambda} = \tilde{\lambda}^T \tag{53}$$

for which either $\mu = -1$ and $\tilde{\Lambda},\tilde{\lambda}$ and skew-symmetric or $\mu = 1$ and $\tilde{\Lambda},\tilde{\lambda}$ are symmetric. Skew-symmetric correlations are called null polarities and since the matrix is of an even order it is generally nonsingular. Null polarities have many interesting properties, especially in relation to the linear complex, Jessop [1903], Busemann and Kelly [1953]. Symmetric correlations are referred to as polarities and are used to establish metrics in Section 3.1. Generally, the only correlations that are employed here subsequently are polarities.

The development of polarities (or more generally correlations) is facilitated by introducing the polarity $\tilde{I}_n$,

$$\tilde{I}_n \tilde{I}_n = \hat{I}_n \tag{54}$$

where $\hat{I}_n$ is the n×n identity collineation. A polarity may be expressed as a product of $\tilde{I}_n$ and a symmetrical collineation k,K,

$$\tilde{\Lambda} = \tilde{I}_4 k = k\tilde{I}_4 \qquad (54)$$

$$\tilde{\lambda} = \tilde{I}_4 K = K\tilde{I}_4 . \qquad (55)$$

In this manner the results obtained for collineations may be applied directly to polarities. The induced polarity of lines corresponding to k and K is respectively

$$\tilde{I}_6 \hat{k} = \hat{k}\tilde{I}_6 \qquad (56)$$

$$\tilde{I}_6 \hat{K} = \hat{K}\tilde{I}_6 . \qquad (57)$$

Analogous to (42), induced polarities have the tetrahedron property. For example, using (56) and (42)

$$\begin{aligned} (\tilde{I}_6 \hat{k})^T \tilde{\Delta} (\tilde{I}_6 \hat{k}) &= \hat{k}^T (\tilde{I}_6 \tilde{\Delta} \tilde{I}_6) \hat{k} \\ &= \hat{k}^T \tilde{\Delta} \hat{k} \\ &= \mu\tilde{\Delta} . \qquad (58) \end{aligned}$$

In Chapter 3 it is shown that polarities may be employed in the development of Euclidean and non-Euclidean geometries using Cayley's conception of the Absolute.

CHAPTER 3

METRICAL GEOMETRY

> Space is another framework which we impose on the world. Whence are the first principles of geometry derived? Are they imposed on us by logic? Lobatschewsky, by inventing non-Euclidean geometries, has shown that this is not the case. Is space revealed to us by our senses? No; for the space revealed to us by our senses is absolutely different from the space of geometry. Is geometry derived from experience? Careful discussion will give the answer--no! We therefore conclude that the principles of geometry are only conventions; but these conventions are not arbitrary, and if transported into another world (which I shall call the non-Euclidean world, and which I shall endeavor to describe), we shall find ourselves compelled to adopt more of them.
>
> In mechanics we shall be led to analogous conclusions, and we shall see that the principles of this science, although more directly based on experience, still share the conventional character of the geometrical postulates. So far, nominalism triumphs; but we now come to the physical sciences, properly so called, and here the scene changes. We meet with hypotheses of another kind, and we full grasp how fruitful they are. No doubt at the outset theories seem unsound, and the history of science shows us how ephemeral they are; but they do not entirely perish, and of each of them some traces still remain. It is these traces which we must try to discover, because in them and in them alone is the true reality. H. Poincare [1905, pp. xxv-xxvi]

Cayley has shown how metrical concepts may be introduced into geometry on a purely projective basis. That is, a figure such as a quadric surface is designated as a fixed reference, the Absolute, and metrical properties, are those properties of figures which take on significance in relation to the Absolute. This is the starting point for the systematic development of metrical geometries in Section 3.1. Metrical relations are developed using projective coordinates and hence the seemingly self-contradictory name of projective metrics. This section simultaneously treats the common properties of hyperbolic, Euclidean and elliptic geometries in a general manner using the notion of an Absolute polarity as an invariant connection of dual elements in space. Using a definition by Clifford, an analytical generalization for determining the pitches and axes of screws is given which apparently may be also found in Buchheim [1884b]. Metrical collineations are defined as those which leave the Absolute invariant and form a subgroup of projective collineations. Norms are then introduced as functions of the Absolute and enables the development of metrical coordinates where the components themselves are significant not just their ratios. Elements of projective space are then assigned a norm of unity although, in the general case, this leads to two sets of metrical coordinates for an element which differ in sign. Elements with a nonunity norm are defined as new types of space elements which have an associated weight or magnitude.

Since the properties of the various metrical geometries vary considerably, Section 3.2 deals exclusively with elliptic geometry. First, the elliptic polarity is introduced which has a close connection with interpreting the coordinates of a space element in terms of dual coordinates and forms a basis for some of the developments in Chapter 4. It is shown that properties which are often erroneously associated with n-dimensional "Euclidean" spaces, such as "orthogonality," are actually properties of elliptic space when homogeneous coordinates are utilized. This is particularly important with respect to some later developments dealing with the "orthogonality" of screws which actually signifies that two screws are elliptic conjugates. Specializing a previous formulation, it is shown that screws in elliptic space have two axes which are elliptic polars and two respective pitches that are reciprocal. These results agree with ones given by Clifford [1873] along with the notion that screws of pitch ±1 have properties of free vectors. Other investigations dealing with screws in elliptic space are given by Buchheim [1884a, 1884b], Cox [1882], Heath [1885] and Ball [1900].

In collineation form, the elliptic polarity of lines and screws is similar in appearance to the important identical relation between ray coordinates and axis coordinates which is a symmetrical correlation. Table 3.2.1 summarizes a number of relations in elliptic geometry which appear very similar to expressions in projective geometry. This exemplifies why it is necessary to have an unambiguous notation to delineate collineations from correlations and ray coordinates from axis coordinates. A systematic development commencing with projective geometry makes it possible to delineate the distinct but similar expressions.

Euclidean geometry is distinguished by the fact that the Absolute polarity is singular and consequently many relations must be approached as limiting cases. As shown in Section 3.3, the singularity introduces an asymmetric character to dual expressions which does not exist for projective or elliptic geometry. The general formulation in Section 3.1 is specialized for an interesting development of the pitch and axis of a screw which is expressible as a linear combination of a unique line and its polar. It is also shown that what is often referred to as the "dual" operator ω, where $\omega^2 = 0$, is merely the Euclidean polarity expressed in form amenable to biquaternions. Based on the Euclidean Absolute, norms are defined which are then used to introduce Euclidean metrical coordinates. Elements of projective space are assigned norms of unity and points are given a unique set of coordinates, unlike planes, lines and screws which have two sets of coordinates that differ in sign. New space elements are defined which have the property of weight or magnitude since their norms are nonunity.

It is typical to study a geometry in terms of examining its group of transformations. However, the group of transformations in Euclidean space always appears to be an entity given a priori from which subsequent geometric properties are then derived. Here, beginning with the general group of projective collineations and Cayley's Absolute, it is shown how the corresponding group of

Euclidean transformations may be deduced as those which leave the Euclidean Absolute invariant. Although the procedure is not complex, it appears to have been previously overlooked.

In Section 3.4 polar and axial vectors in Euclidean space are introduced by way of Klein´s second principle given at the beginning of Section 2.3. Then using Hamilton´s vectors, a polar vector is defined as the difference between two points. By introducing vectors and making the point a more fundamental element in Euclidean space than the plane, the ambiguity of signs for the metrical coordinates of planes, lines and screws is examined. The ambiguity is only resolvable for new space elements that are then introduced namely, plane-sects, geometric couples, line vectors and screw vectors which are all distinguished by a magnitude and an unambiguous associated direction. The ray coordinates and axis coordinates of line vectors and screw vectors are then expressed in a formulation which is typical of modern presentations, especially the ones using dual vectors such as in Brand [1947]. Finally, in application to the area of mechanics, twists and wrenches are introduced along with the formulation of virtual work. When a body is in static equilibrium under impressed wrenches its virtual work vanishes, a property which is shown to be analogous to the projective property of incidence.

Section 3.1 Projective Metrics

> I remark in conclusion, that, in my own point of view, the more systematic course in the present introductory memoir on the geometrical part of the subject of quantics, would have been to ignore altogether the notions of distance and metrical geometry; for the theory in effect is, that the metrical properties of a figure are not the properties of the figure considered per se apart from everything else, but its properties when considered in connexion with another figure, viz the conic termed the Absolute. The original figure might comprise a conic; for instance, we might consider the properties of the figure formed by two or more conics, and we are then in the region of pure descriptive geometry: we pass out of it into metrical geometry by fixing upon a conic of the figure as a standard of reference and calling it the Absolute. Metrical geometry is thus a part of descriptive geometry, and descriptive geometry is all geometry, and reciprocally; and if this be admitted, there is no ground for the consideration, in an introductory memoir, of the special subject of metrical geometry; but as the notions of distance and of metrical geometry could not, without explanation, be thus ignored, it was necessary to refer to them in order to show that they are thus included in descriptive geometry.
> Arthur Cayley [1859, pp. 592]

In the preceding chapter, it has been demonstrated that homogeneous coordinates may be introduced into geometry without recourse to a form of measure, or in other words, a metric. Homogeneous coordinates are well-suited for examining incidence relations which comprises the domain of projective geometry, or "descriptive geometry" as Cayley referred to it. Metrical geometries such as elliptic, Euclidean and hyperbolic may be developed from projective geometry by establishing one or more figures as a fixed reference, which Cayley called the Absolute. In three-dimensional space, various metrical or so-called Cayley-Klein geometries may be developed by defining the Absolute to be a point locus together with a plane envelope of a quadric surface. Projective homogeneous coordinates may be adapted for metrical geometries and in doing so they may also be endowed with the additional property of magnitude, which is a function of the Absolute.

It is most interesting to note, that prior to the landmark paper "A Sixth Memoir on Quantics," Cayley [1859], projective geometry was considered merely to be a somewhat poorer subject in what was then the all-pervasive geometry of Euclid. Initially, not even Cayley recognized the scope of his dictum, "descriptive geometry is all geometry," since he had only considered the geometries known to him at that time, Euclidean and spherical, the former of which he presented for only one and two dimensions. It was left to F. Klein [1871, 1873], some twelve years hence, to demonstrate that the elliptic geometry of Riemann and the hyperbolic geometry of Lobatchewsky and Boylai, the so-called non-Euclidean geometries, may be developed by selecting the Absolute to be respectively an imaginary or real figure.

In this section, metrical geometries are developed in a general format which may then be specialized to yield hyperbolic, Euclidean and elliptic geometries, the latter two of which are investigated in the succeeding sections. Instead of commencing the development here with an Absolute quadric, it is preferred to first establish an Absolute polarity and its adjoint, Coxeter [1965].

A correlation is a linear transformation which maps each element of space into a dual element. A polarity is a symmetric correlation which can be represented by a symmetrical matrix and may be utilized to establish an invariant connection of space between dual elements. This is essentially equivalent to the Cayley-Klein development since there is generally a one-to-one relationship between polarities and quadric surfaces. It should be particularly noted that since metrical geometries are specializations within projective geometry, that metrical geometries must preserve projective properties, in particular, relations of incidence.

The diagonal polarity $\tilde{\pi}$ and its adjoint $\tilde{\Pi}$ where

$$\tilde{\pi} = \begin{bmatrix} \varepsilon & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix} \tag{1}$$

$$\tilde{\Pi} = \begin{bmatrix} 1 & . & . & . \\ . & \varepsilon & . & . \\ . & . & \varepsilon & . \\ . & . & . & \varepsilon \end{bmatrix} \tag{2}$$

are used to establish elliptic, Euclidean and hyperbolic geometries for $\varepsilon = 1,0,-1$ respectively. Since the polarities become singular in the Euclidean case, the appropriate development considers the limiting case $\varepsilon \to 0$.

For the polar relations,

$$X' = \tilde{\Pi}x \tag{3}$$

$$x' = \tilde{\pi} X \tag{4}$$

the plane X' is said to be the polar of point x and the point x' is said to be the pole of plane X. Two points x,y (two planes X,Y) are said to be conjugate when each is incident with the other's polar (pole) and

$$x^T \tilde{\Pi} y = 0 \tag{5}$$

$$X^T \tilde{\pi} Y = 0. \tag{6}$$

Sometimes conjugate points and conjugate planes are referred to respectively as $\tilde{\Pi}$-orthogonal or $\tilde{\pi}$-orthogonal. The Absolute (quadric) is defined as the locus of self-conjugate points and the envelope of self-conjugate planes,

$$x^T \tilde{\Pi} x = 0 \tag{7}$$

$$X^T \tilde{\pi} X = 0. \tag{8}$$

The polarity of points and planes induces a corresponding polarity of lines. In Fig. 3.1.1, the join of points x,y is the line p and the meet of their polar planes X', Y' defines the polar line P'. Alternatively, the meet of planes T,U also defines the same line P and the join of their poles t',u' also defines the same polar line p'. Therefore P' and p', the axis and ray coordinates of the polar line, are given respectively by the nonsquare determinants (see Section 2.2),

$$P' = |\tilde{\Pi}x \ \tilde{\Pi}y| \tag{9}$$

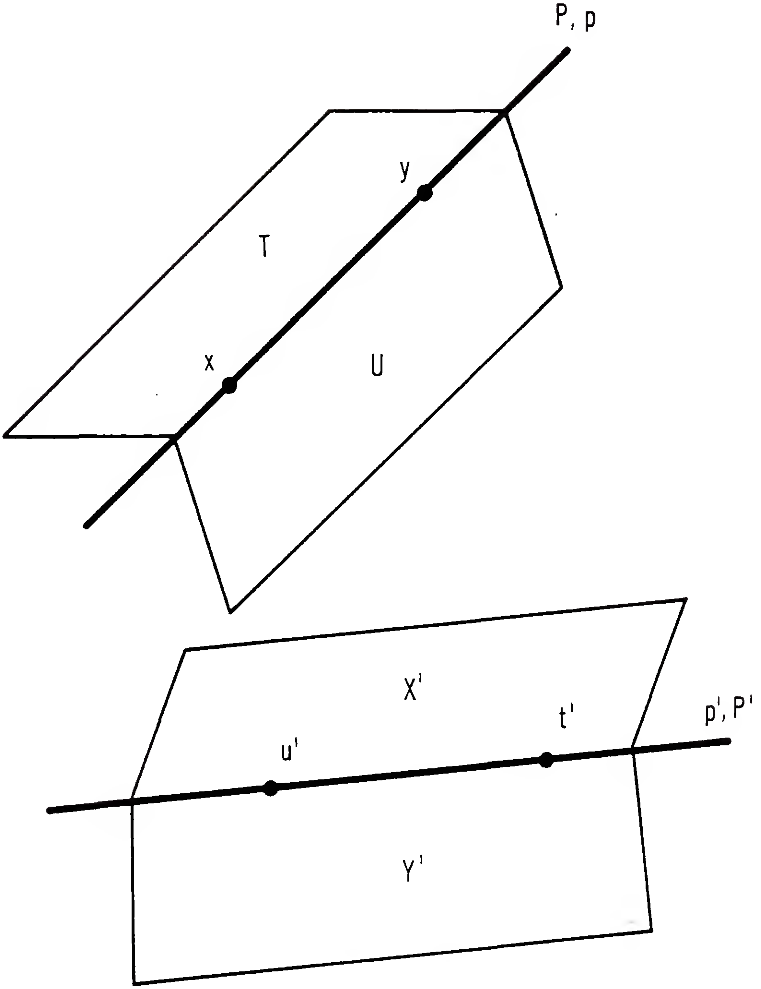

Figure 3.1.1 A pair of polar lines.

$$p' = |\tilde{\pi X} \; \tilde{\pi Y}|. \qquad (10)$$

Substituting, (1), (2) in (9), (10) and expanding yields the induced polar relations

$$P' = \tilde{\Gamma} p \qquad (11)$$

$$p' = \tilde{\gamma} P \qquad (12)$$

where

$$\tilde{\Gamma} = \begin{bmatrix} I_3 & \cdot \\ \cdot & \varepsilon I_3 \end{bmatrix} \qquad (13)$$

$$\tilde{\gamma} = \begin{bmatrix} \varepsilon I_3 & \cdot \\ \cdot & I_3 \end{bmatrix} . \qquad (14)$$

In (13) the common factor ε has been removed which is necessary for the Euclidean case where $\varepsilon \to 0$.

Alternatively, (13) may be derived by substituting the relations between ray and axis coordinates (2.2.48), (2.2.49) in (14) to yield

$$\tilde{\Gamma} = \tilde{\Delta}^T \, \tilde{\gamma} \, \tilde{\Delta}. \qquad (15)$$

Conversely, (15) can be rearranged as

$$\tilde{\gamma} = \tilde{\Delta}^T \, \tilde{\Gamma} \, \tilde{\Delta}. \qquad (16)$$

Since $\tilde{\Gamma}$, $\tilde{\gamma}$ are symmetrical and are adjoints, then to a scalar multiple ε,

$$\tilde{\Gamma}\,\tilde{\gamma} = \hat{I}_6 \tag{17}$$

which may be used with (15),(16) to yield the tetrahedron relationships (see Section 2.3),

$$\tilde{\Gamma}^T\,\tilde{\Delta}\,\tilde{\Gamma} = \tilde{\Delta} \tag{18}$$

$$\tilde{\gamma}^T\,\tilde{\Delta}\,\tilde{\gamma} = \tilde{\Delta}\,. \tag{19}$$

Two lines p,q (P,Q) are said to be conjugate when each is incident with the other's polar line and

$$p^T\,\tilde{\Gamma}\,q = 0 \tag{20}$$

$$P^T\,\tilde{\gamma}\,Q = 0. \tag{21}$$

Alternatively, lines which are conjugate are sometimes referred to as $\tilde{\Gamma}$-orthogonal or $\tilde{\gamma}$-orthogonal. Lines which are self-conjugate are incident with their own polars and their assemblage forms the tangent lines to the Absolute. In line coordinates, the Absolute is given by the quadratic complex, Jessop [1903],

$$p^T\,\tilde{\Gamma}\,p = 0 \tag{22}$$

$$P^T\,\tilde{\gamma}\,P = 0. \tag{23}$$

As previously defined, a linear combination of lines is in general a screw

$$p = \lambda_1 q + \ldots \lambda_n r \quad , \quad P = \lambda_1 Q + \ldots \lambda_n R. \tag{2.2.65}$$

Analogous to (11), (12), the polar of a screw is given by

$$P' = \lambda_1 \tilde{\Gamma} q + \ldots \lambda_n \tilde{\Gamma} r ,$$

$$p' = \lambda_1 \tilde{\gamma} Q + \ldots \lambda_n \tilde{\gamma} R \qquad (24)$$

and two screws which satisfy a bilinear relation of the form (20) or (21) are also said to be conjugate.

In describing screws in elliptic space, Clifford [1873, pp. 193] asserted that a screw can be expressed uniquely as the sum of a line and its polar line and that this polar pair represents the axes of the screw. Here this result is generalized and, using ray coordinates, a screw p is expressed as a linear combination of a unique line q and its polar q'

$$p = \lambda q + \lambda' q' \qquad (25)$$

where it is necessary to determine the scalars λ, λ' and the pair of polar lines q, q'. The line q' may be expressed in ray coordinates by

$$q' = \tilde{\Delta} Q' = \tilde{\Delta} \tilde{\Gamma} q \qquad (26)$$

which is substituted in (25) to yield

$$p = (\lambda \hat{I}_6 + \lambda' \tilde{\Delta} \tilde{\Gamma}) q. \qquad (27)$$

Provided that the matrix in (27) is nonsingular, then using (13) q is easily solved for and

$$q = \frac{1}{(\lambda^2 - \lambda'^2 \varepsilon)} (\lambda \hat{I}_6 - \lambda' \tilde{\Delta} \tilde{\Gamma}) p. \qquad (28)$$

Since q is a line, it must satisfy the identical relation

$$q^T \tilde{\Delta} q = 0 \tag{29}$$

which is used to eliminate q. Substituting (28) in (29) yields, after some rearranging,

$$\frac{1}{(\lambda^2 \ \lambda'^2 \varepsilon)^2} [(p^T \tilde{\Delta} p)\lambda^2 - 2(p^T \tilde{\Gamma} p)\lambda\lambda' + \varepsilon(p^T \tilde{\Delta} p)\lambda'^2] = 0 \tag{30}$$

For $\varepsilon = -1, 0, 1$, (30) may be solved for the ratio $\lambda:\lambda'$ which is then substituted back in (28) to determine q and subsequently q' from (26). By extension of Clifford's definition, the ratios $\lambda':\lambda$ resulting from (30) are called the pitches of the screw with respect to its axes. In the following sections the solution is detailed for both elliptic and Euclidean geometries.

Metrical collineations are defined as those which leave the form of the Absolute invariant. For points and planes, the projective collineations of space (see Section 2.3) are given by

$$y = Kx \quad , \quad Y = kX . \tag{31}$$

In the image space, the Absolute must be expressible in the same form as (7), (8) and

$$y^T \tilde{\Pi} y = 0 \tag{32}$$

$$Y^T \tilde{\pi} Y = 0. \tag{33}$$

Substituting (31) in (32) and (33) yields

$$x^T(K^T\tilde{\Pi}K)x = 0 \tag{34}$$

$$X^T(k^T\tilde{\pi}k)X = 0. \tag{35}$$

Comparing (34), (35) with (7), (8) for general x,X gives

$$K^T\tilde{\Pi}K = \mu\tilde{\Pi} \tag{36}$$

$$k^T\tilde{\pi}k = \mu\tilde{\pi} \tag{37}$$

where μ is a nonzero scalar multiple. Equations (36), (37) express the required conditions for a nonsingular collineation to leave the Absolute invariant. Collineations which satisfy (36) or (37) are referred to as metrical collineations and they form a subgroup within the general group of projective collineations. A projective collineation is expressed using 16 elements and since only the ratios are significant, it is thus determined by 15 independent parameters. However, a metrical collineation must also satisfy (36) or (37), and by symmetry either matrix equation represents a set of 10 nonhomogeneous scalar equations from which μ may be eliminated by considering the 9 ratios of equations. Thus the 15 parameters of a general projective collineation are related by 9 constraint equations to yield 15-9=6 independent parameters for the specification of a metrical collineation.

Briefly, for the induced collineations of lines or screws, let K and k be respectively the ray and axis transformations,

$$q = \hat{K}p \tag{38}$$

$$Q = \hat{k}P. \tag{39}$$

The equations of the Absolute are

$$q^T \tilde{\Gamma} q = p^T(\hat{K}^T \tilde{\Gamma} \hat{K})p = 0 \tag{40}$$

$$Q^T \tilde{\gamma} Q = P^T(\hat{k} \tilde{\gamma} \hat{k})P = 0 \tag{41}$$

and comparing with (22), (23) yields

$$\hat{K}^T \tilde{\Gamma} \hat{K} = \mu\tilde{\Gamma} \tag{42}$$

$$\hat{k}^T \tilde{\gamma} \hat{k} = \mu\tilde{\gamma} \tag{43}$$

which are the conditions for an induced collineation to be a metrical collineation.

The establishment of an Absolute enables the introduction of a projective norm which is useful in the development of metrical coordinates. For points and planes respectively, the norms are given by the scalar functions

$$\|x\| = (x^T \tilde{\Pi} x)^{\frac{1}{2}} \tag{44}$$

$$\|X\| = (X^T \tilde{\pi} X)^{\frac{1}{2}} \tag{45}$$

and for both lines and screws, the norms in terms of ray and axis coordinates are

$$\|p\| = (p^T \tilde{\Gamma} p) \tag{46}$$

$$||P|| = (P^T \tilde{\gamma} P)^{\frac{1}{2}}. \qquad (47)$$

Normed elements are defined as elements whose norms are unity.

Once an Absolute is established, it is possible to remove the restriction from homogeneous coordinates that only the ratios are significant. This transition from projective homogeneous coordinates to metrical homogeneous coordinates is initiated by first normalizing all elements. For example, the projective homogeneous coordinates of a point x are normalized by

$$\frac{x}{||x||} = \frac{x}{(x^T \tilde{\Pi} x)^{\frac{1}{2}}} \quad . \qquad (48)$$

Replacing x with a scalar multiple λx, which also designates the same point in projective coordinates, yields

$$\frac{\lambda x}{||\lambda x||} = \frac{\lambda x}{||\lambda|| \cdot ||x||} = \pm \frac{x}{||x||} \qquad (49)$$

where the norm of the scalar is

$$||\lambda|| = (\lambda^2)^{\frac{1}{2}}. \qquad (50)$$

Therefore in general, the ∞^1 sets of projective homogeneous coordinates which correspond to a single element and differ by a nonzero scalar factor, are transformed by the normalization into two sets of homogeneous metrical coordinates which designate the same element yet differ in sign. By including further constraints, it is possible to resolve this ambiguity of signs completely for hyperbolic geometry, only partially for Euclidean geometry and not at all for elliptic geometry, Busemann and Kelly [1953]. For the case of elliptic geometry, this situation has interesting consequences which are detailed in Section 3.2 together with the Euclidean case in Section 3.3.

It is convenient to sometimes refer to elements of projective space, i.e. points, planes, lines and screws, as unit or unweighted elements since in terms of metrical coordinates they have unity norms. Since all the elements of projective space have a representation in terms of normalized metrical coordinates, it is reasonable to assign a new meaning to metrical coordinates with a nonunity norm. Such coordinates are said to represent weighted elements, Forder [1940], which are purely metrical in nature, i.e. they have no projective representation and are thus a new species of space element. Every weighted element may be represented as a scalar multiple of an unweighted element, e.g. for a weighted point

$$x = \|x\| \cdot \frac{x}{\|x\|} \tag{51}$$

where the weight is simply the norm. Oftentimes the terms weight and magnitude are used synonymously. A common physical example of a weighted element is a point mass.

Metrical collineations have already been defined as a subgroup of collineations which preserve the form of the Absolute and have been formulated using projective coordinates in (36), (37), (42), (43). However, these formulations only define collineations uniquely to a scalar multiple and are thus not suitable when it is desired to employ metrical coordinates with associated weights. For this purpose, it is required that a collineation also preserves the norms of weighted elements. Letting K and k be respectively point and plane collineations, then it is required that the relations

$$K^T \tilde{\Pi} \; K = \tilde{\Pi} \tag{52}$$

$$k^T \tilde{\pi} \; k = \tilde{\pi} \tag{53}$$

be satisfied identically, not just to a scalar factor.

For distinctiveness, a metrical collineation which preserves the norm may be referred to as a unit or unweighted collineation or if the context is clear, simply as a collineation. Weighted collineations do not preserve the norm and are not considered here further. Since a unit collineation may not be multiplied by an arbitrary nonzero scalar factor, it represents 16 parameters. Further, because (52) and (53) are symmetrical, either relation represents 10 independent scalar equations and thus, a metrical collineation is specified by 16-10=6 independent parameters, a result that agrees with the previous one employing projective coordinates.

For the induced unit collineations of lines and screws, let $\hat{K}$ and $\hat{k}$ be respectively the ray and axis transformations then it is required that the relations

$$\hat{K}^T \tilde{\Gamma} \hat{K} = \tilde{\Gamma} \tag{54}$$

$$\hat{k}^T \tilde{\gamma} \hat{k} = \tilde{\gamma} \tag{55}$$

be satisfied identically, not just to a scalar factor. It may be demonstrated that the collineations induced from unit collineations are also unit collineations. First, taking the determinants of (52) - (54) yields

$$|K|^2 = 1 \quad , \quad |k|^2 = 1 \tag{56}$$

$$|\hat{K}|^2 = 1 \quad , \quad |\hat{k}|^2 = 1 \tag{57}$$

which are the necessary and sufficient conditions for the collineations to be unweighted. Using (2.3.2) and (2.3.4), the identical relations (2.3.31) and (2.3.32) may be expressed as

$$\hat{K}^T \tilde{\Delta} \hat{K} = |K| \tilde{\Delta} \tag{58}$$

$$\hat{k}^T \tilde{\Delta} \hat{k} = |k| \tilde{\Delta} \tag{59}$$

and taking the determinants of these equations yields

$$|\hat{K}|^2 = |K|^6 \tag{60}$$

$$|\hat{k}|^2 = |k|^6 \; . \tag{61}$$

Substituting (56) in (60), (61) yields the desired relations (57).

The properties of hyperbolic, Euclidean and elliptic space vary considerably and the preceding general analysis only uncovers relations which are common to all. In the following two sections, specific properties are detailed which, in particular, distinguish elliptic and Euclidean geometries.

Section 3.2 Elliptic Geometry

> Consider any vertical line, and a series of horizontal planes cutting it at right angles. In ordinary or Euclidean geometry these planes intersect on the horizon, which is a straight line infinitely distant. In the geometry of a space of constant positive curvature, or elliptic geometry, the horizon is at a certain finite distance in all directions from the vertical line with which we started; it belongs to that particular line, which is called its polar, and is not the same for all vertical lines. Although it appears to be a great circle when viewed from the neighborhood of its polar, yet if we were to go to it and examine it we should find it straight. Points of it which are in opposite directions from a point on the polar are really identical; and every straight line in this space resembles a circle in being of finite length, so that if we travel far enough along it we shall arrive at our starting point. Every straight line has a polar line, which is the intersection of all planes at right angles to it.
>
> Let us take a very small circle on a sphere, and suppose it to expand, keeping always the same centre. At the beginning the circle will be concave inside and convex outside; but when the expansion has gone on far enough it will become a great circle of the sphere, which is of the same shape on both sides, or is straight so far as the surface of the sphere is concerned. So if in Euclidian space we take a sphere and suppose it to expand, keeping always the same centre, it will continue to be concave inside and convex outside so long as it is finite; but

> when the radius has become infinite, the inside in one direction is the same as the outside in the opposite direction, opposite points being identical; thus the sphere is of the same shape on both sides, or is a plane, viz., the plane at infinity. In elliptic space, just as in geometry on the surface of a sphere, this takes place for a finite length of the radius, not for an infinite length; for every point there is a sphere having its centre at that point, which is also a plane. Or, which is the same thing, every point has a polar plane which is the locus of all points situate at a certain distance from it; this distance is called a quadrant. So also every plane has a certain point, called its pole, which is distant a quadrant from every point in the plane. All lines and planes perpendicular to the plane pass through its pole, and conversely. The polar lines of all lines in the plane pass through its pole, and so do the polar planes of all points in the plane.
>
> When two lines are polars of one another, every point of one is distant a quadrant from every point of the other; hence the polar planes of all points on one pass through the other. Every line which is at right angles to one meets the other, and conversely. W.K. Clifford [1876, pp. 390-391]

In relation to hyperbolic and Euclidean geometries, elliptic geometry has the simplest and most symmetrical properties. The results of the previous section are recounted for elliptic geometry by setting $\varepsilon=1$.

The elliptic polarity for points and planes is

$$\tilde{\Pi}_1 = \tilde{I}_4 \tag{1}$$

$$\tilde{\pi}_1 = \tilde{I}_4 \tag{2}$$

and the polar relations are given by

$$X' = \tilde{\Pi}_1 x = \tilde{I}_4 x \tag{3}$$

$$x' = \tilde{\pi}_1 X = \tilde{I}_4 X. \tag{4}$$

Two points or planes which are elliptic conjugates satisfy respectively

$$x^T \tilde{\Pi}_1 y = x^T \tilde{I}_4 y = 0 \tag{5}$$

$$X^T \tilde{\pi}_1 Y = X^T \tilde{I}_4 Y = 0. \tag{6}$$

The locus of self-conjugate points and the envelope of self-conjugate planes define the elliptic Absolute

$$x^T \tilde{\Pi}_1 x = x^T \tilde{I}_4 x = 0 \tag{7}$$

$$X^T \tilde{\pi}_1 X = X^T \tilde{I}_4 X = 0. \tag{8}$$

For ray and axis coordinates, the induced elliptic polarity is given by

$$\tilde{\Gamma}_1 = \tilde{I}_6 \tag{9}$$

$$\tilde{\gamma}_1 = \tilde{I}_6 \tag{10}$$

and the polar relations are

$$P' = \tilde{\Gamma}_1 p = \tilde{I}_6 p \tag{11}$$

$$p' = \tilde{\gamma}_1 P = \tilde{I}_6 P. \tag{12}$$

Two lines which are elliptic conjugates satisfy

$$p^T \tilde{\Gamma}_1 q = p^T \tilde{I}_6 q = 0 \tag{13}$$

$$P^T \tilde{\gamma}_1 Q = P^T \tilde{I}_6 Q = 0 \tag{14}$$

and the line equations of the elliptic Absolute are

$$p^T \tilde{\Gamma}_1 p = p^T \tilde{I}_6 p = 0 \tag{15}$$

$$P^T \tilde{\gamma}_1 P = P^T \tilde{I}_6 P = 0. \tag{16}$$

By linearity, the elliptic polarity of screws is given by

$$P' = \lambda_1 \tilde{\Gamma}_1 q + \ldots \lambda_n \tilde{\Gamma}_1 r = \lambda_1 \tilde{I}_6 q + \ldots \lambda_n \tilde{I}_6 r \tag{17}$$

$$p' = \lambda_1 \tilde{\gamma}_1 Q + \ldots \lambda_n \tilde{\gamma}_1 R = \lambda_1 \tilde{I}_6 Q + \ldots \lambda_n \tilde{I}_6 R \tag{18}$$

and two screws which satisfy (13) or (14) are also said to be elliptic conjugates.

For points, planes, lines and screws the elliptic polarity is specified by a correlation which is the identity matrix, either $\tilde{I}_4$ in (1), (2) or $\tilde{I}_6$ in (9), (10). Thus, the elliptic polar of an element is determined by interpreting its coordinates in terms of dual coordinates, (3), (4), (11), (12), (17), (18). Two elements which are elliptic conjugates (5), (6), (13), (14), are also said to be either elliptic-orthogonal or more specifically prefixed by the polarity such as for points, $\tilde{\Pi}_1$-orthogonal or $\tilde{I}_4$-orthogonal. From the equations of the Absolute, (7), (8), (15), (16) it is clear that the points, planes and lines which form the

Absolute are not real and thus it is often referred to as an imaginary or virtual quadric surface. The elements forming the Absolute are said to be elliptic self-conjugate or alternatively, elliptic-isotropic.

For lines and screws, it is very important to carefully distinguish projective relations from the metrical elliptic relations which are similar in appearance due to the elementary form of the elliptic polarity. The projective correlation between ray coordinates and axis coordinates

$$P = \tilde{\Delta} p \tag{2.2.60}$$

$$p = \tilde{\Delta} P \tag{2.2.61}$$

may be substituted in (11), (12) to express the elliptic polarity as a collineation

$$P' = \tilde{I}_6\ p = \tilde{I}_6\ \tilde{\Delta}\ P = \hat{\Delta} P \tag{19}$$

$$p' = \tilde{I}_6\ P = \tilde{I}_6\ \tilde{\Delta}\ p = \hat{\Delta} p \tag{20}$$

where the product of correlations $\tilde{I}_6\tilde{\Delta}$ yields the collineation

$$\tilde{\Delta} = \begin{bmatrix} \cdot & I_3 \\ I_3 & \cdot \end{bmatrix} . \tag{21}$$

Further, substituting (2.2.60) and (2.2.61) in (13), (14) yields alternative formulations for elliptic-orthogonality,

$$p^T \tilde{I}_6 q = P^T \tilde{\Delta} \tilde{I}_6 q = P^T \hat{\Delta} q = 0 \qquad (22)$$

$$P^T \tilde{I}_6 Q = p^T \tilde{\Delta} \tilde{I}_6 Q = p^T \hat{\Delta} Q = 0. \qquad (23)$$

Table 3.2.1 summarizes the projective relations along side the elliptic relations which are very similar in form. As throughout, a tilda is used to signify that a transformation is a correlation, e.g. $\tilde{I}_6$, whereas a caret denotes a collineation, e.g. $\hat{I}_6$. In the table, each expression contains either a correlation or a collineation and its correspondent contains the other. Further, for each pair of expressions, one of the two screws (or lines) which correspond are in dual coordinates while the other two are in the same coordinates. Without a clear notational distinction between correlations and collineations and between ray and axis coordinates these relations are easily confused and misinterpreted.

It was first discovered by Clifford [1873], that in elliptic space a screw p can in general be expressed as a linear combination of a unique line q and its polar q',

$$p = \lambda q + \lambda' q' \qquad (3.1.25)$$

both of which are called the axes of the screw. The ratios $\lambda':\lambda$ and $\lambda:\lambda'$ are respectively the pitches of the screw with respect to q and q'. Setting $\varepsilon=1$ in the general formulations (3.1.27), (3.1.28) and (3.1.30) yields

$$p = (\lambda \hat{I}_6 + \lambda' \hat{\Delta}) q \qquad (24)$$

Table 3.2.1 Projective and metrical relations that are similar in appearance

Projective	Metrical
1. Transformations of ray and axis coordinates	1. Transformations of the elliptic polarity
$P = \tilde{\Delta} p$	$p' = \hat{\Delta} p$
$p = \tilde{\Delta} P$	$P' = \hat{\Delta} P$
$p = \hat{I}_6 \; p$	$P' = \tilde{I}_6 \; p$
$P = \hat{I}_6 \; P$	$p' = \tilde{I}_6 \; P$
2. Reciprocity	2. Elliptic-orthogonality
$p^T \tilde{\Delta} q = 0$	$P^T \hat{\Delta} q = 0$
$P^T \tilde{\Delta} Q = 0$	$p^T \hat{\Delta} Q = 0$
$p^T \hat{I}_6 Q = 0$	$P^T \tilde{I}_6 Q = 0$
$P^T \hat{I}_6 q = 0$	$p^T \tilde{I}_6 q = 0$

$$q = \frac{1}{(\lambda^2 - \lambda'^2)} \; [\lambda \hat{I}_6 - \lambda' \hat{\Delta}]p \tag{25}$$

$$\frac{1}{(\lambda^2 - \lambda'^2)} \; [(p^T \tilde{\Delta}\, p)\lambda^2 - 2(p^T \tilde{I}_6\, p)\lambda\lambda' + (p^T \tilde{\Delta}\, p)\lambda'^2] = 0. \tag{26}$$

Assuming first that $\lambda^2 \neq \lambda'^2$ and $p^T \tilde{\Delta}\, p \neq 0$ then (26) may be expressed by

$$\lambda^2 - 2b\lambda\lambda' + \lambda'^2 = 0 \tag{27}$$

where

$$b = \frac{p^T \tilde{I}_6\, p}{p^T \tilde{\Delta}\, p} \; . \tag{28}$$

The solutions to (27) are

$$\frac{\lambda'}{\lambda} = b + (b^2 - 1)^{\frac{1}{2}} \, , \; b - (b^2 - 1)^{\frac{1}{2}} \tag{29}$$

or equivalently

$$\frac{\lambda'}{\lambda} = \mu, \; \frac{1}{\mu} \tag{30}$$

where

$$\mu = b + (b^2 - 1)^{\frac{1}{2}} \; . \tag{31}$$

Substituting in turn the roots (30) back in (25) yields the corresponding pair of axes

$$\lambda q_\mu = \frac{1}{(1 - \mu^2)} \; [\hat{I}_6 - \mu \hat{\Delta}]p \tag{32}$$

$$\lambda q_{1/\mu} = \frac{\mu\hat{\Delta}}{(1-\mu^2)} [\hat{I}_6 - \mu\hat{\Delta}]p \tag{33}$$

and since

$$(\mu\hat{\Delta})q_\mu = q_{1/\mu} \tag{34}$$

the two axes are elliptic polars where the subscripts denote the associated pitches.

When $p^T \tilde{\Delta} p = 0$, p is a line and assuming that $p^T \tilde{I}_6 p \neq 0$ in (26), then either $\lambda=0$ or $\lambda'=0$ and thus from (3.1.25), respectively either $p=\lambda' q'$ or $p=\lambda q$.

The exceptional case occurs when $\lambda^2=\lambda'^2$ and thus the pitch of the screw is either +1 or -1. The form of this screw p may be determined by first taking the polar of (3.1.25),

$$p' = \lambda q' + \lambda' q \tag{35}$$

and then subtracting and adding in turn (3.1.25) and (35) yields

$$(p'-p) = (\lambda-\lambda')(q'-q) \tag{36}$$

$$(p'+p) = (\lambda+\lambda')(q'+q). \tag{37}$$

When the pitch is +1 then (36) yields

$$p=p' \quad \text{or} \quad p=\tilde{\Delta}p \tag{38}$$

and thus the first and last three components of p are the same. When the pitch is -1 then (37) yields

$$p = -p' \quad \text{or } p = -\hat{\Delta} p$$

and the first and last three components are the same except that they have opposite signs. As a consequence of these special forms, screws of pitch +1 may be expressed in an infinite number of ways as the sum of a line and its polar. The axes of such screws are thus indeterminant and for this reason Clifford referred to them as vectors, a right vector when the pitch is +1 and a left vector when the pitch is -1. Clifford also made the observation that a screw p, of any pitch, may be expressed as the sum of a right vector and left vector uniquely,

$$p = \frac{(p+p')}{2} + \frac{(p-p')}{2} \; . \tag{39}$$

Vectors are closely related to right and left parallel lines in elliptic space which are also called paratactics, Sommerville [1929], or Clifford parallels, but are not developed here.

The transition from projective coordinates to elliptic metrical coordinates is made by first introducing elliptic norms. By setting $\varepsilon=1$ in (3.1.44) - (3.1.47), the elliptic norms for points, planes and lines or screws become

$$\| x \|_1 = (x^T \tilde{I}_4 \, x)^{\frac{1}{2}} \tag{40}$$

$$\|x\|_1 = (x^T \tilde{I}_4 x)^{\frac{1}{2}} \tag{41}$$

$$\|p\|_1 = (p^T \tilde{I}_6 p)^{\frac{1}{2}} \tag{42}$$

$$\|P\|_1 = (P^T \tilde{I}_6 P)^{\frac{1}{2}}. \tag{43}$$

As discussed in Section 3.1, by normalizing projective coordinates, elements in projective space become unit or unweighted elements in terms of metrical coordinates. In general, an unweighted element has two representations in metrical coordinates which differ only in sign. For elliptic space, there is no additional criteria that can be imposed which enables a definitive selection of sign without ambiguity. The resolution to this predicament is merely to associate the pair of coordinates with each element, e.g. x and -x for a unit point. For the elliptic plane this has an interesting consequence since it enables modeling in Euclidean space by a unit sphere. That is, a point on the elliptic plane is modeled either by a pair antipodal points on a unit sphere or equivalently by a diametrical line through these points. A line on the elliptic plane is represented by a great circle. The analogy in space is somewhat more involved, however Clifford's description given at the beginning of this section is highly suggestive.

Elliptic collineations are defined as the subgroup of projective collineations which leave the elliptic Absolute invariant. In terms of metrical coordinates, the unweighted collineations for points, planes and lines or screws satisfy the relations given by setting $\varepsilon=1$ in (3.1.52) - (3.1.55),

$$K^T \tilde{I}_4 K = \tilde{I}_4 \tag{44}$$

$$k^T \tilde{I}_4 k = \tilde{I}_4 \tag{45}$$

$$\hat{K}^T \tilde{I}_6 \hat{K} = \tilde{I}_6 \tag{46}$$

$$\hat{k}^T \tilde{I}_6 \hat{k} = \tilde{I}_6. \tag{47}$$

Analogous to the convention for elements, a collineation has two distinct representations which differ by sign, e.g. K and -K for point collineations, see Busemann and Kelly [1953]. From the above relations it is easy to deduce that the determinant of an elliptic collineation is +1 or -1. However, both representations of a collineation have the same determinant since the matrices are of an even order, e.g. $|-K| = (-1)^4|K| = |K|$. Further, it is readily deduced from (44) - (47) that the inverse of an elliptic collineation is equal to its transpose, e.g. $K^{-1} = K^T$. In the literature such matrices are usually referred to as orthogonal or orthonormal. However, for distinctiveness elliptic collineations are referred to here as elliptic-orthogonal or prefixed with the polarity such as $\tilde{I}_4$-orthogonal.

Section 3.3 Euclidean Geometry

> We shall find throughout this period, that almost every important proposition, though misleading in its obvious interpretation, has nevertheless, when rightly interpreted, a wide philosophical bearing. So it is with the work of Cayley, the pioneer of the projective method.

The projective formula for angles, in Euclidean Geometry, was first obtained by Laguerre, in 1853. This formula had, however, a perfectly Euclidean character, and it was left for Cayley to generalize it so as to include both angles and distances in Euclidean and non-Euclidean systems alike.

Cayley was, to the last, a staunch supporter of Euclidean space, though he believed that non-Euclidean Geometries could be applied, within Euclidean space, by a change in the definition of distance. He has thus, in spite of his Euclidean orthodoxy, provided the believers in the possibility of non-Euclidean spaces with one of their most powerful weapons. In his "Sixth Memoir upon Quantics" (1859), he set himself the task of "establishing the notion of distance upon purely descriptive principles." He showed that, with the ordinary notion of distance, it can be rendered projective by reference to the circular points and the line at infinity, and that the same is true of angles. Not content with this, he suggested a new definition of distance, as the inverse sine or cosine of a certain function of the coordinates; with this definition, the properties usually known as metrical become projective properties, having reference to a certain conic, called by Cayley the Absolute. (The circular points are, analytically, a degenerate conic, so that ordinary Geometry forms a particular case of the above.) He proves that, when the Absolute is an imaginary conic, the Geometry so obtained for two dimensions is spherical Geometry. The correspondence with Lobatchewsky, in the case where the Absolute is real, is not worked out; indeed there is, throughout, no evidence of acquaintance with non-Euclidean systems. The importance of the memoir, to Cayley, lies entirely in its proof that metrical is only a branch of descriptive Geometry.

The connection of Cayley's Theory of Distance with Metageometry was first pointed out by Klein. Klein showed in detail that, if the Absolute be real, we get Lobatchewsky's (hyperbolic) system; if it be imaginary, we get either spherical Geometry or a new system, analogous to that of Helmholtz, called by Klein elliptic; if the Absolute be an imaginary point-pair, we get parabolic Geometry, and if, in particular, the point-pair be the circular points, we get ordinary Euclid. . . .

> Since these systems are all obtained from a Euclidean plane, by a mere alteration in the definition of distance, Cayley and Klein tend to regard the whole question as one, not of the nature of space, but of the definition of distance. Since this definition, on their view, is perfectly arbitrary, the philosophical problem vanishes--Euclidean space is left in undisputed possession, and the only problem remaining is one of convention and mathematical convenience. Bertrand Russell [1897, pp. 29-30]

In contrast to elliptic space, for which the elementary form of the polarity yields many symmetrical relations amongst dual elements, the singular nature of the Euclidean Absolute yields highly unsymmetrical dualistic properties. As a consequence, various general formulas which may be specialized for elliptic and hyperbolic geometry, do not appear applicable to Euclidean geometry unless they are treated as limiting cases where $\varepsilon \to 0$. One example is the formula for the distance between two points which yields an indeterminant result unless an infinitesmal device is utilized as in Klein [1908]. However, for most of the developments here it is possible to set $\varepsilon = 0$ in the general formulations of Section 3.1.

The Euclidean polarity for points and planes is given by

$$\tilde{\Pi}_0 = \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} \tag{1}$$

$$\tilde{\pi}_0 = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} . \tag{2}$$

(As throughout, zero elements in an array are often denoted by periods and I_1 is used here in place of 1 for greater symmetry.) The polar relations are

$$X' = \tilde{\Pi}_0 x = \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} x = \begin{bmatrix} x_0 \\ \cdot \end{bmatrix} \tag{3}$$

$$x' = \tilde{\pi}_0 X = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} X = \begin{bmatrix} \cdot \\ \underline{X} \end{bmatrix} \tag{4}$$

where the notation $\underline{X}$ and $\underline{x}$ is introduced,

$$\underline{X} = [X_1 \; X_2 \; X_3]^T \tag{5}$$

$$\underline{x} = [x_1 \; x_2 \; x_3]^T. \tag{6}$$

From (3), for a point x not on the plane at infinity, $x \neq [0 \; x_1 \; x_2 \; x_3]^T$, its polar plane X' is in fact the plane at infinity, $X' = [x_0 \; 0 \; 0 \; 0]^T$. For a point x on the plane at infinity, $x = [0 \; x_1 \; x_2 \; x_3]^T$, its polar plane X' is indeterminate since all of its components vanish, $X' = [0 \; 0 \; 0 \; 0]^T$.

From (4), for a plane X which itself is not the plane at infinity, $X \neq [X_0 \; 0 \; 0 \; 0]^T$, then its pole x' is a point on the plane at infinity, $x' = [0 \; X_1 \; X_2 \; X_3]^T$. For X itself being the plane at infinity, $X = [X_0 \; 0 \; 0 \; 0]^T$, its pole x' is indeterminate $x' = [0 \; 0 \; 0 \; 0]^T$.

Two points or planes which are Euclidean conjugates satisfy respectively,

$$x^T \tilde{\Pi}_0 \, y = x^T \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} y = x_0 \, y_0 = 0 \qquad (7)$$

$$X^T \tilde{\pi}_0 \, Y = X^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} Y = \underline{X}^T \, \underline{Y} = 0. \qquad (8)$$

In (7) two points are Euclidean conjugates if either lies on the plane at infinity. In (8), two planes which are Euclidean conjugates are said to be Euclidean-orthogonal and thus meet at right angles.

The angles of self-conjugate points and the envelope of self-conjugate planes define the Euclidean Absolute

$$x^T \tilde{\Pi}_0 \, x = x^T \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} x = x_0^2 = 0 \qquad (9)$$

$$X^T \tilde{\pi}_0 \, X = X^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} X = \underline{X}^T \, \underline{X} = \underline{X}_1^2 + \underline{X}_2^2 + \underline{X}_3^2. \qquad (10)$$

Equation (9) represents the plane at infinity taken twice which is a rank one quadric locus. Equation (10) represents what is referred to as the imaginary spherical circle since it is the intersection of every sphere with the plane at infinity and further, because it is a rank three quadric envelope it represents a conic, see Klein [1908], Sommerville [1934]. Thus unlike elliptic geometry, the locus and envelope of the Euclidean Absolute are two distinct figures which is a consequence of the Euclidean polarity being singular.

In terms of ray and axis coordinates, the induced Euclidean polarity is respectively given by,

$$\tilde{\Gamma}_0 = \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} \tag{11}$$

$$\tilde{\gamma}_0 = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} \tag{12}$$

and the corresponding polar relations are

$$P' = \tilde{\Gamma}_0\, p = \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{0} \end{bmatrix} \tag{13}$$

$$p' = \tilde{\gamma}_0\, P = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{P} \end{bmatrix} \tag{14}$$

where the 3×1 arrays $\underline{p}$, $\underline{p}_0$, $\underline{P}$, $\underline{P}_0$, $\underline{0}$ are introduced,

$$\underline{p} = [p_{01}\ p_{02}\ p_{03}]^T\ ,\ \underline{p}_0 = [p_{23}\ p_{31}\ p_{12}]^T \tag{15}$$

$$\underline{P} = [P_{23}\ P_{31}\ P_{12}]^T\ ,\ \underline{P}_0 = [P_{01}\ P_{02}\ P_{03}]^T \tag{16}$$

$$\underline{0} = [0\ 0\ 0]^T. \tag{17}$$

Considering firstly (14), it may be shown that the polar of any line P not on the plane at infinity is in fact a line on the plane at infinity, $p' = [0\ 0\ 0\ P_{23}\ P_{31}\ P_{12}]^T$. Let the line P be the meet of two finite planes, $P = |XY|$. The polar line p' is equivalent to the join of the two poles of X and Y which are points on the plane at infinity, $p' = |X'\ Y'|$ and thus represents a line on the plane at infinity.

This type of reasoning is not effective for (13). Let line p be the join of two points p = |xy| and the polars of x and y are both the plane at infinity. Since the meet of any plane with itself vanishes identically, this does not yield the line given by (13). It is not clear why this synthetic argument fails. However, it should be noted that (13) was derived from (3.1.13) where it was necessary to delete a common factor ε before ε was set to zero to yield (13). Alternatively, (13) may be derived directly from (14) be multiplying throughout by $\tilde{\Delta}$,

$$\tilde{\Delta} p' = \tilde{\Delta} \begin{bmatrix} \underline{0} \\ \hline \underline{p} \end{bmatrix} \qquad (18)$$

which yields the desired result

$$P' = \begin{bmatrix} \underline{p} \\ \hline \underline{0} \end{bmatrix}. \qquad (19)$$

It also follows from both (13) and (14) that the polar of any line on the plane at infinity is indeterminate since its coordinates vanish. Thus the Euclidean polar of the Euclidean polar of a line or screw vanishes. This is exactly equivalent to a double application of the Euclidean polar operator ω, where $\omega^2=0$, to a rotor or motor (line or screw) and was invented by Clifford [1873] in the development of biquaternions. For elliptic space, Clifford used the polar operator ω, where $\omega^2=1$, which is equivalent to (3.2.19) and (3.2.20). Clifford did not discuss hyperbolic geometry in this context but the corresponding polar operator has the property $\omega^2 = -1$. An elegant development of biquaternions and the three related polar operators is given by Veblen and Young [1917].

Two lines or screws which are Euclidean conjugates satisfy

$$p^T \tilde{\Gamma}_0 q = p^T \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} q = \underline{p}^T \underline{q} = 0 \tag{20}$$

$$P^T \tilde{\gamma}_0 Q = P^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} Q = \underline{P}^T \underline{Q} = 0 \tag{21}$$

and are said to be Euclidean-orthogonal and are at right angles.

The line envelopes of the Euclidean Absolute are

$$p^T \tilde{\Gamma}_0 p = p^T \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} p = \underline{p}^T \underline{p} = 0 \tag{22}$$

$$P^T \tilde{\gamma}_0 P = P^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} P = \underline{P}^T \underline{P} = 0 \tag{23}$$

which both represent the plane at infinity taken twice and the imaginary spherical circle. Equations (22) and (23) may be shown equivalent by using the transformation between ray and axis coordinates and (3.1.16),

$$\begin{aligned} p^T \tilde{\Gamma}_0 p &= (\tilde{\Delta} P)^T \tilde{\Gamma}_0 (\tilde{\Delta} P) \\ &= P^T (\tilde{\Delta} \tilde{\Gamma}_0 \tilde{\Delta}) P \\ &= P^T \tilde{\gamma}_0 P. \end{aligned} \tag{24}$$

In general, the points, planes and lines which form the Absolute are said to be Euclidean self-conjugate or alternately Euclidean isotropic.

By linear principles, the Euclidean polarity of screws may be expressed by

$$P' = \lambda_1 \tilde{\Gamma}_0 q + \ . \ . \ . \ \lambda_n \tilde{\Gamma}_0 r \tag{25}$$

$$p' = \lambda_1 \tilde{\gamma}_0 Q + \ . \ . \ . \ \lambda_n \tilde{\gamma}_0 R. \tag{26}$$

In Euclidean space, the pitch of a screw is determined by expressing it as a linear combination of a unique line and its polar.

$$p = \lambda q + \lambda' q' \tag{3.1.25}$$

where the ratio $\lambda':\lambda$ is the pitch of the screw. Setting $\varepsilon=0$ in the general formulation (3.1.30) yields

$$\frac{1}{\lambda^3}\left[(p^T \tilde{\Delta}\, p)\lambda - 2(p^T \tilde{\Gamma}_0\, p)\lambda'\right] = 0. \tag{27}$$

For $\lambda \neq 0$, the pitch is unique and is given by

$$h = \frac{\lambda'}{\lambda} = \tfrac{1}{2}\,\frac{p^T \tilde{\Delta}\, p}{p^T \tilde{\Gamma}_0\, p}\ . \tag{28}$$

The corresponding axis of the screw is also unique and is given by substituting $\varepsilon=0$ in (3.1.28)

$$\lambda q = \begin{bmatrix} I_3 & \cdot \\ -hI_3 & I_3 \end{bmatrix} p \tag{29}$$

which is equivalent to

$$\lambda q = \begin{bmatrix} \underline{p} \\ \underline{p}_0 - h\underline{p} \end{bmatrix} . \tag{30}$$

When $\lambda=0$, then it is necessary to use (3.1.27) with $\varepsilon=0$ to yield

$$p = \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} = \lambda' \begin{bmatrix} \underline{0} \\ \underline{q} \end{bmatrix} , \ \lambda' \neq 0 \tag{31}$$

and thus p is a line on the plane at infinity and has infinite pitch

$$h = \frac{\lambda'}{\lambda} = \frac{\lambda'}{0} = \infty . \tag{32}$$

In terms of axis coordinates, for $h \neq \infty$ the pitch is given by,

$$h = \tfrac{1}{2} \frac{P^T \tilde{\Delta} P}{P^T \tilde{\gamma}_0 P} \tag{33}$$

and the axis of the screw is

$$\lambda Q = \begin{bmatrix} \underline{Q}_0 - h\underline{Q} \\ \underline{Q} \end{bmatrix} . \tag{34}$$

When $h=\infty$ then the screw is a line on the plane at infinity and

$$P = \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} = \lambda' \begin{bmatrix} \underline{Q} \\ \underline{0} \end{bmatrix} , \ \lambda' \neq 0 . \tag{35}$$

Euclidean metrical coordinates are developed from projective coordinates by first introducing Euclidean norms. Setting $\varepsilon=0$ in (3.1.44) - (3.1.47) yields the Euclidean norms for points, planes and lines or screws

$$\|x\|_0 = (x^T \tilde{\Pi}_0 x)^{\frac{1}{2}} = (x_0^2)^{\frac{1}{2}} \tag{36}$$

$$\|X\|_0 = (X^T \tilde{\pi}_0 X)^{\frac{1}{2}} = (\underline{X}^T \underline{X})^{\frac{1}{2}} \tag{37}$$

$$\|p\|_0 = (p^T \tilde{\Gamma}_0 p)^{\frac{1}{2}} = (\underline{p}^T \underline{p})^{\frac{1}{2}} \tag{38}$$

$$\|P\|_0 = (P^T \tilde{\gamma}_0 P)^{\frac{1}{2}} = (\underline{P}^T \underline{P})^{\frac{1}{2}}. \tag{39}$$

Projective coordinates are normalized by dividing through with the norm. Elements in projective space become unit or unweighted elements in terms of metrical coordinates. Generally, the normalization process leads to two sets of unweighted coordinates for an element which differ in sign. However for finite points, $x_0 \neq 0$, the normalization may be made unique

$$\frac{x}{\sigma(x_0)\,\|x\|} = \frac{x}{x_0} \tag{40}$$

where $\sigma(x_0)$ is the sign of x_0. Thus the first coordinate of a normalized finite point is always positive unity, $x_0=1$.

When a point lies on the plane at infinity, $x_0=0$, it is not possible to normalize it in this manner. It is however possible to adopt a suitable normalization by first noting that a point on the plane at infinity may be regarded as the pole of a finite plane

$$x' = \tilde{\pi}_0 X. \tag{41}$$

Therefore, it is useful to define an unweighted point on the plane at infinity as the pole as an unweighted plane

$$\tilde{\pi}_0 \frac{X}{\|X\|} = \frac{\tilde{\pi}_0 X}{(X^T \tilde{\pi}_0 X)^{\frac{1}{2}}} = \frac{\tilde{\pi}_0 X}{((X^T \tilde{\pi}_0)\tilde{\pi}_0(\tilde{\pi}_0 X))^{\frac{1}{2}}} = \frac{x'}{(x'^T \tilde{\pi}_0 x')^{\frac{1}{2}}}$$

$$= \frac{x'}{(\underline{x}'^T \underline{x}')^{\frac{1}{2}}} \tag{42}$$

where

$$(\tilde{\pi}_0)^3 = \tilde{\pi}_0. \tag{43}$$

In distinction to finite points, points on the plane at infinity do not have a unique representation since like plane coordinates, the sign is not determinate.

The only instance for which the plane norm (37) vanishes is for the plane at infinity which is the polar of every finite point. Thus in a normalized form it may be considered to be the polar of any finite unweighted point such that its coordinates are $[1\ 0\ 0\ 0]^T$.

The norm of a line or screw (38), (39) vanishes only for a line on the plane at infinity which is the polar of a finite line or screw,

$$P' = \tilde{\Gamma}_0 \, p \tag{44}$$

$$p' = \tilde{\gamma}_0 \, P . \tag{45}$$

Thus an unweighted line on the plane at infinity may be defined as the polar of a finite unweighted line or screw, which in analogy with the result for points (42), is given for axis coordinates by

$$\frac{\tilde{\Gamma}_0 \, p}{\| p \|} = \frac{P'}{(P'^T \tilde{\Gamma}_0 P')^{\frac{1}{2}}} = \frac{1}{(\underline{P}_0'^T \underline{P}_0')^{\frac{1}{2}}} \begin{bmatrix} \underline{P}_0' \\ \underline{0} \end{bmatrix} \tag{46}$$

and is given for ray coordinates by

$$\frac{\tilde{\gamma}_0 \, P}{\| P \|} = \frac{p'}{(p'^T \tilde{\gamma}_0 p')^{\frac{1}{2}}} = \frac{1}{(\underline{p}_0'^T \underline{p}_0')^{\frac{1}{2}}} \begin{bmatrix} \underline{0} \\ \underline{p}_0 \end{bmatrix} . \tag{47}$$

Using incidence relations in Section 2.3, the relationship between projective point and plane collineation was deduced as

$$k = \mu K^{-T} \tag{48}$$

where for simplicity, the arbitrary nonzero scalar μ was set to unity, (2.3.9). Unweighted metrical collineation preserve the norms of elements and as such their determinants are ±1. However, since points in Euclidean space may be assigned a unique representation, it becomes necessary for the relations between K and k to explicitly include the signs of their determinants.

For the unweighted point collineation

$$K = [A\ B\ C\ D]^T\ ,\ |K|^2 = 1 \qquad (49)$$

the corresponding unweighted plane collineation is defined by replacing each element of K by its cofactor

$$k = [|B\ C\ D|\ \ |C\ A\ D|\ \ |A\ B\ D|\ \ |B\ A\ C|]^T,\ |k|^2 = 1. \qquad (50)$$

Using the definition of a matrix inverse in terms of cofactors then

$$k = |K| K^{-T}\ ,\ |K|^2 = 1. \qquad (51)$$

For $|K| = -1$, this result differs from (48) with $\mu=1$. Taking the determinant of (51) yields

$$|k| = |K|^3 = |K| \qquad (52)$$

and then inverting (51) using (52) gives

$$K = |k| k^{-T}\ ,\ |k|^2 = 1. \qquad (53)$$

The unweighted line collineations corresponding to (51) and (53) are derived by forming the ray and axis coordinates for the edges of the corresponding tetrahedron as in Section 2.3 and

$$\hat{k} = |K|^{12}\ \hat{K}^{-T}\ ,\ |K|^2 = 1 \qquad (54)$$

$$\hat{K} = |k|^{12} \hat{k}^{-T} \quad , \quad |k|^{2} = 1 \tag{55}$$

which are clearly equivalent to the previously used form

$$\hat{k} = \hat{K}^{-T} \; . \tag{2.2.30}$$

Further, using (52) and (2.2.30) it is easily shown that the identities (2.3.31) - (2.3.35) remain applicable.

Euclidean collineations are defined as the subgroup of projective collineations which leaves the Euclidean Absolute invariant. Using metrical coordinates, the unweighted collineations for points, planes and lines or screws must satisfy the relations (3.1.52) - (3.1.55) with $\varepsilon=0$,

$$K^{T} \tilde{\Pi}_0 K = \tilde{\Pi}_0 \quad , \quad K^{T} \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} K = \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} \tag{56}$$

$$k^{T} \tilde{\pi}_0 k = \pi_0 \quad , \quad k^{T} \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} k = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} \tag{57}$$

$$\hat{K} \tilde{\Gamma}_0 \hat{K} = \tilde{\Gamma}_0 \quad , \quad \hat{K}^{T} \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} \hat{K} = \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} \tag{58}$$

$$\hat{k} \tilde{\gamma}_0 \hat{k} = \tilde{\gamma}_0 \quad , \quad \hat{k}^{T} \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} \hat{k} = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} \; . \tag{59}$$

In order to deduce the form of the point and plane collineations, it is useful to express K as

$$K = \begin{bmatrix} f & g \\ \underline{t} & h \end{bmatrix} \tag{60}$$

where h is a 3×3 array and $\underline{t}$ is 3×1. Substitution of (60) in (56) yields

$$\begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} f^T f & f^T g \\ g^T f & g^T g \end{bmatrix} \qquad (61)$$

for which it is easily determined that

$$I_1 = f^T f \text{ or } f = \pm I_1 \qquad (62)$$

$$0 = g^T g \text{ or } g = 0. \qquad (63)$$

The remaining elements of K are determined by substituting the identity $k = K^{-T}$ in (57) and rearranging to yield

$$\tilde{\pi}_0 = K\, \tilde{\pi}_0\, K^T. \qquad (64)$$

Then substitution of (60) in (64) yields

$$\begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} = \begin{bmatrix} gg^T & gh^T \\ hg^T & hh^T \end{bmatrix}. \qquad (65)$$

from which it is deduced that

$$hh^T = I_3 \text{ or } h^T = h^{-1}. \qquad (66)$$

The orthonormal matrices of (66) may be divided into two distinct sets which are characterized by the determinant being either +1 or -1. Since the matrix is of an odd degree, a member of one set may be transformed into a member of the other by multiplying with -1 or equivalently $-I_3$. Thus, h may be expressed in the form

$$h = \sigma_3 e = e\sigma_3 \qquad (67)$$

where e is orthonormal with a positive determinant

$$ee^T = I_3 \ , \ |e| = 1 \qquad (68)$$

and where σ_n may assume either of the two values

$$\sigma_n = I_n \text{ or } \sigma_n = -I_n. \qquad (69)$$

Therefore using (62), (63), (66), (67) in (60) yields the form of the Euclidean point collineation

$$K = \begin{bmatrix} I_1 & \cdot \\ \underline{t} & e\sigma_3 \end{bmatrix}. \qquad (70)$$

In (70) the positive value for f in (62) has been selected since it is necessary for K to transform a normalized unweighted point into the same. Additionally, since the equations that determine K are independent of $\underline{t}$, then it is clear that $\underline{t}$ may assume arbitrary values.

It may be easily verified that Euclidean collineations (70), for a group. Further, three distinct subgroups of Euclidean collineations may be identified by expressing (70) in the form

$$K = K_t \, K_e \, K_\sigma$$

$$= \begin{bmatrix} I_1 & \cdot \\ \underline{t} & I_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & \sigma_3 \end{bmatrix} . \qquad (71)$$

The first subgroup K_t represents the group of Euclidean translations which is in general characterized by the property that no finite points remain fixed (the exception is for $\underline{t} = \underline{0}$ and thus $K_t = I_4$). The second subgroup K_e represents the group of Euclidean rotations which are all characterized by the property that the origin always remains fixed. Additionally, for each rotation there is a unique line which remains pointwise invariant and is called the axis of rotation. Again, the exceptional case is for $K_e = I_4$ in which all lines are pointwise invariant. The third subgroup K_σ consists of only two collineations for which one is the identity $\sigma_3 = I_3$, and the other is a reflection through the origin $\sigma_3 = -I_3$. In such a reflection, spheres centered about the origin remain invariant since antipodal points are mapped into each other.

In Euclidean space, rigid body collineations or equivalently rigid body motions are a subgroup of the Euclidean collineations (63) and consist of only translations K_t and rotations K_e. Reflection through the origin K_σ must be excluded since it is a discontinuous transformation in relation to the identity collineation, see Klein [1908].

The corresponding Euclidean collineation of planes are determined from (71) using the relation between unweighted point and plane collineations (51) and are tabulated in Table 3.3.1 along with the point collineations.

The induced line or screw collineations for ray coordinates may be determined directly from (70) by forming the six edges of the corresponding tetrahedron as in (2.3.21). Then using various determinant identities, the general induced collineation may be expressed as the product of a translation, a rotation and a reflection through the origin. However, it is preferred to use the factored form of (71) and apply an identity for the product of two general collineations K and J,

$$\begin{aligned} |KJx\ KJy| &= \hat{K}|Jx\ Jy| \\ &= \hat{K}\ \hat{J}\ |xy|. \end{aligned} \tag{72}$$

Thus applying (72) to (71) yields

$$\begin{aligned} \hat{K} &= \hat{K}_t\ \hat{K}_e\ \hat{K}_\sigma \\ &= \begin{bmatrix} I_3 & \cdot \\ t^* & I_3 \end{bmatrix} \begin{bmatrix} e & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} I_3 & \cdot \\ \cdot & \sigma_3 \end{bmatrix} \end{aligned} \tag{73}$$

where t* is the skew-symmetric array

$$t^* = \begin{bmatrix} \cdot & -t_3 & t_2 \\ t_3 & \cdot & -t_1 \\ -t_2 & t_1 & \cdot \end{bmatrix}. \tag{74}$$

Table 3.3.1 Euclidean collineations for point and plane coordinates

$$K = \begin{bmatrix} I_1 & \cdot \\ \underline{t} & e\sigma_3 \end{bmatrix} = \begin{bmatrix} I_1 & \cdot \\ \underline{t} & I_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & \sigma_3 \end{bmatrix} = K_t \; K_e \; K_\sigma$$

$$K^{-1} = \begin{bmatrix} I_1 & \cdot \\ -\sigma_3 e^T \underline{t} & \sigma_3 e^T \end{bmatrix} = \begin{bmatrix} I_1 & \cdot \\ \cdot & \sigma_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e^T \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ -\underline{t} & I_3 \end{bmatrix} = K_\sigma^{-1} \; K_e^{-1} \; K_t^{-1}$$

$$k = \begin{bmatrix} \sigma_1 & -\underline{t}^T e \\ \cdot & e \end{bmatrix} = \begin{bmatrix} I_1 & -\underline{t}^T \\ \cdot & I_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} \sigma_1 & \cdot \\ \cdot & I_3 \end{bmatrix} = k_t \; k_e \; k_\sigma$$

$$k^{-1} = \begin{bmatrix} \sigma_1 & \sigma_1 \underline{t}^T \\ \cdot & e^T \end{bmatrix} = \begin{bmatrix} \sigma_1 & \cdot \\ \cdot & I_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e^T \end{bmatrix} \begin{bmatrix} I_1 & \underline{t}^T \\ \cdot & I_3 \end{bmatrix} = k_\sigma^{-1} \; k_e^{-1} \; k_t^{-1}$$

The induced collineations for axis coordinates may be easily determined using $\hat{k} = \hat{K}^{-T}$, (2.2.30). The Euclidean collineations for the ray and axis coordinates of lines or screws are presented in Table 3.3.2.

For completeness, Euclidean measurement formulas are briefly summarized. It was Cayley who first formulated the measurement of distances and angles on a purely projective basis with his introduction of the Absolute. Later, Klein [1871, 1873] reformulated Cayley's work in terms of the logarithms of a cross ratio which is a function of the Absolute. He also extended the results to include elliptic and hyperbolic geometries.

Since the derivations of the Euclidean measurement formulas are quite lengthy, they are presented without detail. The distance between two unweighted points x and y is given by

$$((x - y)^T (x - y))^{\frac{1}{2}} = ((\underline{x} - \underline{y})^T (\underline{x} - \underline{y}))^{\frac{1}{2}} \tag{75}$$

and the angle θ between two unweighted or weighted planes X and Y is defined by

$$\cos\theta = \frac{X^T \tilde{\pi}_0 Y}{(X^T \tilde{\pi}_0 X)^{\frac{1}{2}} (Y^T \tilde{\pi}_0 Y)^{\frac{1}{2}}} = \frac{\underline{X}^T \underline{Y}}{(\underline{X}^T \underline{X})^{\frac{1}{2}} (\underline{Y}^T \underline{Y})^{\frac{1}{2}}} . \tag{76}$$

It may be easily verified that both (75) and (76) are invariant expressions with respect to Euclidean collineations.

Table 3.3.2 Euclidean collineations for ray and axis coordinates

$$\hat{K} = \begin{bmatrix} e\sigma_3 & \cdot \\ t^* e\sigma_3 & e \end{bmatrix} = \begin{bmatrix} I_3 & \cdot \\ t^* & I_3 \end{bmatrix}\begin{bmatrix} e & \cdot \\ \cdot & e \end{bmatrix}\begin{bmatrix} \sigma_3 & \cdot \\ \cdot & I_3 \end{bmatrix} = \hat{K}_t \, \hat{K}_e \, \hat{K}_\sigma$$

$$\hat{K}^{-1} = \begin{bmatrix} \sigma_3 e^T & \cdot \\ -e^T t^* & e^T \end{bmatrix} = \begin{bmatrix} \sigma_3 & \cdot \\ \cdot & I_3 \end{bmatrix}\begin{bmatrix} e^T & \cdot \\ \cdot & e^T \end{bmatrix}\begin{bmatrix} I_3 & \cdot \\ -t^* & I_3 \end{bmatrix} = \hat{K}_\sigma^{-1} \, \hat{K}_e^{-1} \, \hat{K}_t^{-1}$$

$$\hat{k} = \begin{bmatrix} e\sigma_3 & t^* e \\ \cdot & e \end{bmatrix} = \begin{bmatrix} I_3 & t^* \\ \cdot & I_3 \end{bmatrix}\begin{bmatrix} e & \cdot \\ \cdot & e \end{bmatrix}\begin{bmatrix} \sigma_3 & \cdot \\ \cdot & I_3 \end{bmatrix} = \hat{k}_t \, \hat{k}_e \, \hat{k}_\sigma$$

$$\hat{k}^{-1} = \begin{bmatrix} \sigma_3 e^T & -\sigma_3 e^T t^* \\ \cdot & e^T \end{bmatrix} = \begin{bmatrix} \sigma_3 & \cdot \\ \cdot & I_3 \end{bmatrix}\begin{bmatrix} e^T & \cdot \\ \cdot & e^T \end{bmatrix}\begin{bmatrix} I_3 & -t^* \\ \cdot & I_3 \end{bmatrix} = \hat{k}_\sigma^{-1} \, \hat{k}_e^{-1} \, \hat{k}_t^{-1}$$

Section 3.4 Vector Formulations

> I do not wish to end this discussion without pointing out, with emphasis, that, for our general standpoint, the questions of ordinary vector analysis constitute only a chapter out of a profusion of more general problems. For example, line-bound vectors, restricted plane-magnitudes, screws, and systems of forces are, strictly speaking, not considered in vector analysis. For a real understanding of the operations of vector algebra themselves, however, it is actually necessary to take a broader view of them. Only then does the principle which inheres in them, namely, that of defining geometric magnitudes according to their behavior under the various kinds of transformation of rectangular coordinates, find full expression.
> Felix Klein [1908, pp. 51]

In the previous sections it was shown that the Absolute may be used to establish metrical norms and that the elements of projective space may be assigned a norm of unity. The norm was also used to define new elements which are strictly metrical in nature and are referred to as weighted elements since equivalently, their weight, magnitude or norm is nonunity. Because of the quadratic form of the Absolute, it was demonstrated in general that an element has two sets of coordinates which differ in sign.

For Euclidean space, it was possible to alter the process of normalization such that finite points could be assigned a unique set of coordinates. This was not however possible for planes, lines, screws and points at infinity. The reason for this asymmetry is because the Euclidean Absolute is derived from a singular polarity for which the point locus and plane envelopes are distinct figures. There is however a manner in which the ambiguity of signs may be resolved for new species of space elements which are closely related to the previous ones.

In the development of projective geometry, the point and plane are the fundamental elements of space and since they are considered entirely equivalent in significance, the duality of projective space complete. However in Euclidean space the asymmetry of the Absolute, and consequently the unique representation of points, alters this balance in significance of dual elements such that it becomes proper to elevate the status of the point by designating it as a more fundamental element than the plane. That is, points may be first used to define planes and then in turn they may both be used to define other space elements. Thus all elements are made to depend either directly or indirectly on point definitions.

The ambiguity of signs is resolved by defining the property of directional sense or simply direction. This leads to the establishment of new space elements, namely free vectors, line vectors and screw vectors, which are also called motors. These elements include both the properties of magnitude and direction in their definition.

The approach presented here is to first introduce free vectors, or simply vectors, and to initially describe their properties in terms of the difference of two points. Since the formulation and interpretation of expressions in Euclidean geometry is often facilitated by the introduction of free vectors, they are then subsequently utilized in defining the new space elements. The goal here is not to reproduce ordinary vector analysis, but to demonstrate the logical evolution of space elements with particular emphasis on line vectors and screw vectors. As such, a familiarity of basic vector concepts in Euclidean geometry is assumed and only the portions necessary for further developments in the sequal are derived. A rather elegant and detailed account of the introduction of vector elements into geometry has been presented by Klein [1908], portions of which are utilized here.

Klein's second principle, which is given at the beginning of Section 2.3, may be applied in the particular form: under a transformation of space, transformed coordinates which can be expressed as a linear homogeneous function of only their previous coordinates, define a geometrical space element or configuration. Using the group of Euclidean collineations given in Table 3.3.1, two new space elements may be initially identified, polar free vectors and axial free vectors. The general results are first presented to emphasize the many interrelationships and a detailed analysis follows. Under the general collineation of Euclidean space, the corresponding collineations of three distinct polar vectors are given by the expressions

$$\underline{w} = e\ \sigma_3\ \underline{x} \tag{1}$$

$$\underline{q} = e\ \sigma_3\ \underline{p} \tag{2}$$

$$\underline{M}_0 = e\ \sigma_3\ \underline{P}_0 \tag{3}$$

and the general collineations of three distinct axial vectors are given by

$$\underline{W} = e\ \underline{X} \tag{4}$$

$$\underline{Q} = e\ \underline{P} \tag{5}$$

$$\underline{m}_0 = e\ \underline{p}_0 \tag{6}$$

where both types of vectors are expressed as 3×1 arrays and are denoted by an underbar as in (3.3.5-6) and (3.3.15-17). It is important to note in (1) - (6) that polar and axial vectors both remain invariant under translations of space. The distinguishing feature between the two types is that axial vectors are also invariant with respect to a reflection through the origin whereas polar vectors change in sign. Under the group of Euclidean rigid body collineations where reflection through the origin is excluded, clearly this distinction vanishes.

Prior to subsequent development, it is useful to first introduce some standard vector notations, identities and definitions. The scalar product (dot product), vector product (cross product) and scalar triple product are defined respectively by

$$\underline{\alpha} \cdot \underline{\beta} = \underline{\alpha}^T\ \underline{\beta} \tag{7}$$

$$\underline{\alpha} \times \underline{\beta} = |\underline{\alpha}\ \underline{\beta}| = [|\underline{\alpha}_2\ \underline{\beta}_3|\ \ |\underline{\alpha}_3\ \underline{\beta}_1|\ \ |\underline{\alpha}_1\ \underline{\beta}_2|]^T \tag{8}$$

$$\underline{\alpha} \times \underline{\beta} \cdot \underline{\gamma} = |\underline{\alpha}\ \underline{\beta}\ \underline{\gamma}| \tag{9}$$

where $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ may be polar or axial vectors. The following identities are frequently used,

$$\underline{\alpha} \times \underline{\alpha} = \underline{0} \tag{10}$$

$$\underline{\alpha} \times \underline{\beta} = - \underline{\beta} \times \underline{\alpha} \tag{11}$$

$$\underline{\alpha} \times (\underline{\beta} \times \underline{\gamma}) = (\underline{\gamma} \times \underline{\beta}) \times \underline{\alpha} = (\underline{\alpha} \cdot \underline{\gamma})\underline{\beta} - (\underline{\alpha} \cdot \underline{\beta})\underline{\gamma}. \tag{12}$$

The norm, weight or magnitude of a free vector is defined by

$$\| \underline{\alpha} \| = (\underline{\alpha} \cdot \underline{\alpha})^{\frac{1}{2}} \tag{13}$$

and the angle θ measured from $\underline{\alpha}$ to $\underline{\beta}$ in a right-hand sense about a normal $\underline{\gamma}$ is defined by

$$\cos \theta = \frac{\underline{\alpha} \cdot \underline{\beta}}{\| \underline{\alpha} \| \cdot \| \underline{\beta} \|} \tag{14}$$

$$\sin \theta = \frac{\underline{\alpha} \times \underline{\beta} \cdot \underline{\gamma}}{\| \underline{\alpha} \| \cdot \| \underline{\beta} \| \cdot \| \underline{\gamma} \|} . \tag{15}$$

Further, by selecting $\underline{\gamma}$ such that $\sin \theta \geq 0$, then

$$\underline{\alpha} \times \underline{\beta} = \| \underline{\alpha} \| \cdot \| \underline{\beta} \| \cdot \sin \theta \, \frac{\underline{\gamma}}{\| \underline{\gamma} \|} , \quad 0 \leq \theta \leq \pi. \tag{16}$$

Hamilton [1866] defined a vector $\underline{x}$ as a line segment with a direction and he conceived of it as the difference of two finite points

$$\begin{bmatrix} 0 \\ \underline{x} \end{bmatrix} = \begin{bmatrix} 1 \\ \underline{z} \end{bmatrix} - \begin{bmatrix} 1 \\ \underline{y} \end{bmatrix} \tag{17}$$

which may be graphically represented as an arrow pointing from y to z. In (17), it is clear that the difference of two points has the same form as a weighted point on the plane at infinity and it thus transforms in an identical manner,

$$\begin{bmatrix} w_0 \\ \underline{w} \end{bmatrix} = K \begin{bmatrix} 0 \\ \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ e \ \sigma_3 \ \underline{x} \end{bmatrix} \tag{18}$$

where the last three equations yield (1). Polar vectors therefore have the same properties as weighted points at infinity. In particular, a translation of space leaves points at infinity unchanged and thus the vector $\underline{x}$, which is graphically portrayed as an arrow, is not bound between the two points y and z but is free to translate without altering its geometric significance. Hence the name, free vector. It should also be noted that the magnitude of a vector (13) is equivalent to the norm of a weighted point at infinity which appears in (3.3.42) for normalization.

There is a unique polar vector associated with each finite point which is formed as the difference of the point with the origin and is referred to here as the vector of the point. Clearly, its components are equivalent to the last three coordinates of the point and its magnitude is the distance from the point to the origin. Conversely, a unique point may be associated with each vector, namely that point whose difference with the origin yields the vector, and may thus be referred to as the point of the vector. This latter conception is usually called a position vector which is somewhat of a misnomer since it actually designates a finite point (by its nonhomogeneous coordinates) and is not a vector quantity.

The ray coordinates of a weighted line may be expressed as the join of two finite points

$$p = |xy| = \begin{bmatrix} \underline{y} - \underline{x} \\ \underline{x} \times \underline{y} \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} \qquad (19)$$

and the general collineation of p is given by

$$\begin{bmatrix} \underline{q} \\ \underline{q}_0 \end{bmatrix} = \hat{K}_p = \begin{bmatrix} e\, \sigma_3\, \underline{p} \\ t^* \; e\, \sigma_3\, \underline{p} + e\, \underline{p}_0 \end{bmatrix} \qquad (20)$$

The first three equations yield (2) and thus $\underline{p}$ is a polar vector which is also evident from (19) where essentially $\underline{p}$ is formed as the difference of points x and y. The weight of p is given by (3.3.38) which is identical to the magnitude of vector $\underline{p}$. Therefore by associating a free vector with a line or equivalently by associating a sense of direction with a weighted line, a new space elements is established which is referred to as a line vector, a line-bound vector or a rotor as Clifford [1873] named it. Line vectors, or more precisely polar line vectors in this case, are detailed subsequently along with screw vectors.

The ray coordinates of a weighted line at infinity may be expressed as the join of the points at infinity

$$p = \begin{vmatrix} 0 & 0 \\ \underline{x} & \underline{y} \end{vmatrix} = \begin{bmatrix} \underline{0} \\ \underline{x} \times \underline{y} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{p}_0 \end{bmatrix} \qquad (21)$$

and the general collineation of the line is given by

$$\hat{K} \begin{bmatrix} \underline{0} \\ \underline{p}_0 \end{bmatrix} = \begin{bmatrix} \underline{0} \\ e\ \underline{p}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \underline{m}_0 \end{bmatrix}. \qquad (22)$$

The last three equations represent the transformation of an axial vector as expressed by (6). Thus a free vector may be associated with a weighted line at infinity and it is therefore endowed with a directional sense.

The sum of two parallel polar line vectors with equal magnitudes and opposite senses may be expressed by

$$\begin{bmatrix} \underline{q} \\ \underline{q}_0 \end{bmatrix} + \begin{bmatrix} -\underline{q} \\ (\underline{p}_0 - \underline{q}_0) \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{p}_0 \end{bmatrix} , \quad -\underline{q} \cdot \underline{p}_0 = 0 \qquad (23)$$

and is defined as an axial free line vector. Since the free line vector has the same form as a directed line at infinity and also transforms by (22), they both have the same properties. In particular, a free line vector is invariant with respect to translations and reflections through the origin. This situation is quite analogous to the correspondence between a weighted point at infinity and the difference of two finite points which are each free vectors.

The correspondence of a vector with a weighted plane is first established from the general collineation

$$\begin{bmatrix} W_0 \\ \underline{W} \end{bmatrix} = k \begin{bmatrix} X_0 \\ \underline{X} \end{bmatrix} = \begin{bmatrix} \sigma_1 \; X_0 - \underline{t}^T \; e \; \underline{X} \\ e \; \underline{X} \end{bmatrix} . \qquad (24)$$

The last three equations represent the transformation of an axial vector $\underline{X}$ as given by (4) and the magnitude of $\underline{X}$ is the same as the norm of the plane given by (3.3.37). However, a precise interpretation of the correspondence of the free vector with the plane is not possible from (24) alone. As stated in the beginning of this section, it is necessary to express the coordinates of a plane in terms of point coordinates.

Let the weighted plane be expressed as the join of three noncollinear points u, v, w and

$$X = |u \;\; v \;\; w| \;\; = |u \;\; (v-u) \;\; (w-u)|$$

$$= \begin{bmatrix} \underline{u} \cdot (\underline{v}-\underline{u}) \times (\underline{w}-\underline{u}) \\ -(\underline{v}-\underline{u}) \times (\underline{w}-\underline{u}) \end{bmatrix} = \begin{bmatrix} -\underline{u} \cdot \underline{X} \\ \underline{X} \end{bmatrix} \qquad (25)$$

where $(\underline{v}-\underline{u})$ and $(\underline{w}-\underline{u})$ are two distinct polar vectors which are parallel to the plane. Using the cosine formula (14) with the identical relations

$$\underline{X} \cdot (\underline{v}-\underline{u}) = 0 \quad , \quad \underline{X} \cdot (\underline{w}-\underline{u}) = 0 \qquad (26)$$

then $\underline{X}$ is perpendicular to both polar vectors and is thus a normal vector to the plane. The magnitude of $\underline{X}$ is determined from (16) by

$$\|X\| = \|\underline{v}-\underline{u}\| \cdot \|\underline{w}-\underline{u}\| \cdot \|\sin\theta\| \tag{27}$$

Further, from (16) and (25), $\underline{X}$ points in the direction which is opposite to a normal defined by a right-handed rotation about the point u, v, w consecutively. It may also be deduced from (25) that $X_0/\|X\|$ is the directed perpendicular distance between the origin and X measured in a direction opposite to the normal $\underline{X}$.

A plane with an associated vector $\underline{X}$ has the properties of magnitude and direction and is referred to here as a plane-sect (plane-section) although plane-magnitude or leaf are more common designations. Under a translation of space a plane-sect transforms by

$$k_t \begin{bmatrix} X_0 \\ \underline{X} \end{bmatrix} = \begin{bmatrix} X_0 - \underline{t}^T \underline{X} \\ \underline{X} \end{bmatrix} . \tag{28}$$

For translations $\underline{t}$ which are perpendicular to the normal $\underline{X}$ then $\underline{t}^T \underline{X} = 0$ in (28) and the plane-sect transforms into itself. Thus a plane-sect is not fixed to any three specific points u, v, w, but is able to assume any position on the plane. Further, from (25) it is easily shown that

$$X_0 = |\underline{u}\ \underline{v}\ \underline{w}| \tag{29}$$

which represents six times the signed volume of the tetrahedron formed by the origin and points u, v, w and is thus proportional to the signed area of triangle u, v, w. Therefore a plane-sect may be represented by any three points on the plane which form an equivalent signed area since the height of the tetrahedron remains unaltered.

The axis coordinates of a line vector may be expressed as the meet of two nonparallel plane-sects X and Y,

$$P = |XY| = \begin{bmatrix} X_0\,\underline{Y} - Y_0\,\underline{X} \\ \underline{X} \times \underline{Y} \end{bmatrix} = \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} \qquad (30)$$

and the general collineation of P is given by

$$\begin{bmatrix} \underline{Q}_0 \\ \underline{Q} \end{bmatrix} = \hat{k}P = \begin{bmatrix} t^{*}\; e\,\underline{P} + e\,\sigma_3\,P_0 \\ e\,\underline{P} \end{bmatrix} . \qquad (31)$$

The last three equations yield (5) and thus $\underline{P}$ is an axial vector. It should be noted the line vectors p and P transform differently under a reflection through the origin. For the purpose of distinction they may be called polar and axial line vectors respectively although simply line vector suffices when the context is clear.

The vector $\underline{P}$ is at right angles to both plane-sect normals since using (30),

$$\underline{P}\cdot\underline{X} = 0 \quad , \quad \underline{P}\cdot\underline{Y} = 0. \qquad (32)$$

Further the magnitude of the line or line vector is given by (3.3.39) which is equivalent to the magnitude of vector $\underline{P}$.

The axis coordinates of an oriented and weighted line at infinity may be expressed as the meet of two distinct and parallel plane-sects

$$P = |XY| = \begin{bmatrix} X_0 \underline{Y} - Y_0 \underline{X} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{P}_0 \\ \underline{0} \end{bmatrix} \qquad (33)$$

Its general collineation is given by

$$k \begin{bmatrix} \underline{P}_0 \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \sigma_3 \underline{P}_0 \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{M}_0 \\ \underline{0} \end{bmatrix} \qquad (34)$$

of which the first three equations yield the transformation of the polar vector in (3).

The sum of two parallel axial line vectors with equal magnitudes and opposite senses may be expressed by

$$\begin{bmatrix} \underline{Q}_0 \\ \underline{Q} \end{bmatrix} + \begin{bmatrix} (\underline{P}_0 - \underline{Q}_0) \\ -\underline{Q} \end{bmatrix} = \begin{bmatrix} \underline{P}_0 \\ \underline{0} \end{bmatrix}, \quad -\underline{Q} \cdot \underline{P}_0 = 0 \qquad (35)$$

and is defined as a polar free line vector. Since the free line vector has the same form as a directed line at infinity and also transforms by (34), they both have similar properties. Most importantly, they are invariant with respect to translations of space.

The representation of line vectors is now examined in further detail. Working first with ray coordinates, (19) may be expressed in the alternative form

$$p = |xy| = \begin{bmatrix} \underline{y}-\underline{x} \\ \underline{x} \times (\underline{y}-\underline{x}) \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{x} \times \underline{p} \end{bmatrix} . \tag{36}$$

Since $\underline{p}$ indicates the direction of the line, then any finite point r on the line is given by

$$r = \begin{bmatrix} 1 \\ \underline{y} + \lambda \underline{p} \end{bmatrix} \tag{37}$$

where λ is any scalar. (Note that in previous sections r has been used to designate a line. This change in notation enables a consistency with the following section.) Solving (37) for y and substituting it in (36) yields

$$p = \begin{bmatrix} \underline{p} \\ \underline{r} \times \underline{p} \end{bmatrix} . \tag{38}$$

Thus for a line vector written in the form (38), r may represent any point on the line.

The point r_n on p whose vector is normal to $\underline{p}$ satisfies

$$\underline{p} \cdot \underline{r}_n = 0. \tag{39}$$

Expanding the vector product

$$\underline{p} \times \underline{p}_0 = \underline{p} \times (\underline{r} \times \underline{p}) = (\underline{p} \cdot p)\underline{r} - (\underline{p} \cdot \underline{r})\underline{p} \tag{40}$$

and using (39) yields the vector of the point r_n,

$$\underline{r}_n = \frac{\underline{p} \times \underline{p}_0}{\underline{p} \cdot \underline{p}} \quad . \tag{41}$$

Unlike free vectors, line vectors are generally altered in significance under translations of space. Using (38), then under a translation p is transformed by

$$\hat{K}_t \begin{bmatrix} \underline{p} \\ \underline{r} \times \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{p} \\ (\underline{t} + \underline{r}) \times \underline{p} \end{bmatrix} \tag{42}$$

where identically

$$t^* \; \underline{p} = \underline{t} \times \underline{p} \; . \tag{43}$$

There is one case of exception when p remains invariant and occurs when the translation $\underline{t}$ is parallel or antiparallel to $\underline{p}$ and the line vector transforms into itself. For this reason a line vector is often conceptualized as an arrow which constrained to the line and hence the term line-bound vector. Since the arrow may slide along the line without altering its significance, the term sliding vector is also used.

The properties of line vectors may also be determined using axis coordinates. Forming the meet of two plane-sects, the line vector is represented by

$$P = |XY| = \begin{bmatrix} X_0 \; \underline{Y} - Y_0 \; \underline{X} \\ \underline{X} \times \underline{Y} \end{bmatrix} = \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} \quad \text{repeated,} \tag{30}$$

and as previously described, $\underline{P}$ indicates the direction and magnitude of the line vector.

In order to interpret $\underline{P}_0$, it is useful to express it in an alternative form analogous to (38). Firstly, it is necessary to demonstrate that there exists a point on the line r_n whose vector $\underline{r}_n$ is normal to the line and is given by

$$\underline{r}_n = \frac{\underline{P} \times \underline{P}_0}{\underline{P} \cdot \underline{P}} . \tag{44}$$

Since it is easily verified that

$$\underline{r}_n \cdot \underline{P} = 0 \tag{45}$$

then $\underline{r}_n$ is clearly normal to $\underline{P}$. Next, for

$$r_n = \begin{bmatrix} 1 \\ \underline{r}_n \end{bmatrix} \tag{46}$$

it may also be verified using (30), (44) that

$$X^T r_n = 0 \quad \text{and} \quad Y^T r_n = 0 \tag{47}$$

and thus r_n is incident to both plane-sects X,Y and consequently r_n is incident to P. Further, since

$$\underline{r}_n \times \underline{P} = \underline{P}_0 \tag{48}$$

is an identical relation it follows that

$$P = \begin{bmatrix} \underline{r}_n \times \underline{P} \\ \underline{P} \end{bmatrix} \tag{49}$$

or more generally,

$$P = \begin{bmatrix} \underline{r} \times \underline{P} \\ \underline{P} \end{bmatrix} \qquad (50)$$

where r is any point on the line vector and

$$r = \begin{bmatrix} 1 \\ \underline{r}_n + \lambda\sigma_3 \ \underline{P} \end{bmatrix} \qquad (51)$$

where λ may assume any value and σ_3 is included such that $\underline{r}$ is a polar vector.

Under a translation of space P is transformed by

$$\hat{k}_t \begin{bmatrix} \underline{r} \times \underline{P} \\ \underline{P} \end{bmatrix} = \begin{bmatrix} (\underline{t} + \underline{r}) \times \underline{P} \\ \underline{P} \end{bmatrix} \qquad (52)$$

and thus when the translation $\underline{t}$ is parallel or antiparallel to $\underline{P}$, the line vector is transformed into itself.

A screw has been previously defined as a linear combination of lines. It has been demonstrated in Section 3.3 that in Euclidean space a screw may be expressed in one way as linear combination of a unique line and its polar where the line is said to be the axis of the screw and where the ratio of scalar multipliers determines the pitch. This leads directly to Ball's [1900] definition of a screw as a line with an associated scalar called the pitch. By the definition of a norm, it was also possible to attach a weight to the screw which is equivalent to the weight of its line or axis. A screw vector may be defined as a weighted screw with an associated free vector whose magnitude is equal to that of the screw. Alternatively, a screw vector may be defined as a line vector with an associated pitch. Screw vectors were referred to as motors by Clifford [1873].

Analogous to a screw, a screw vector can be expressed as a linear combination of a line and its polar in either ray coordinates

$$p = \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{r} \times \underline{p} \end{bmatrix} + h \begin{bmatrix} \underline{0} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{r} \times \underline{p} + h\underline{p} \end{bmatrix} \qquad (53)$$

or in axis coordinates

$$P = \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} = \begin{bmatrix} \underline{r} \times \underline{P} \\ \underline{P} \end{bmatrix} + h \begin{bmatrix} \underline{P} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{r} \times \underline{P} + h\underline{P} \\ \underline{P} \end{bmatrix} \qquad (54)$$

where h is the pitch of the screw vector. From (3.3.28) and (3.3.33) the pitch may be expressed in a vector formulation using ray coordinates

$$h = \frac{\underline{p} \cdot \underline{p}_0}{\underline{p} \cdot \underline{p}} \qquad (55)$$

or axis coordinates

$$h = \frac{\underline{P} \cdot \underline{P}_0}{\underline{P} \cdot \underline{P}} \quad . \qquad (56)$$

Further, it is easily demonstrated that the vector $\underline{r}_n$, the vector from the origin to the screw vector which is normal to its axis, is given by the same formulation as those for line vectors, (41) and (44).

Finally, it is desirable to briefly explain the roles of line vectors and screw vectors in mechanics. It is well known that a force has the attributes of a line vector. Following Plucker's [1866] original conventions, a force is expressed in terms of ray coordinates by

$$f = \begin{bmatrix} \underline{f} \\ \underline{r} \times \underline{f} \end{bmatrix} = \begin{bmatrix} \underline{f} \\ \underline{f}_0 \end{bmatrix} \tag{57}$$

where $\underline{f}$ expresses the direction and magnitude of the force and $\underline{r} \times \underline{f}$ is said to be the moment of the force about the origin.

The couple is a physical example of what has been previously referred to as a free line vector. A couple may be expressed as the sum of two distinct forces with equal magnitudes which act in opposite directions

$$m = \begin{bmatrix} \underline{f} \\ \underline{r}_1 \times \underline{f} \end{bmatrix} + \begin{bmatrix} -\underline{f} \\ \underline{r}_2 \times (-\mathrm{f}) \end{bmatrix} = \begin{bmatrix} \underline{0} \\ (\underline{r}_1 - \underline{r}_2) \times \underline{f} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{m}_0 \end{bmatrix} . \tag{58}$$

According to Poinsot's theorem (see Ball [1900]), a system of forces f_i and couples m_i may be reduced to or is equivalent to a single force and a moment parallel to the force

$$w = \Sigma f_i + \Sigma m_i \tag{59}$$

which Ball has called a wrench on a screw or more simply a wrench, w, which may be expressed using (53) by

$$w = \begin{bmatrix} \underline{w} \\ \underline{r} \times \underline{w} + h\underline{w} \end{bmatrix} = \begin{bmatrix} \underline{w} \\ \underline{r} \times \underline{w} \end{bmatrix} + h \begin{bmatrix} \underline{0} \\ \underline{w} \end{bmatrix} = \begin{bmatrix} \underline{w} \\ \underline{w}_0 \end{bmatrix} . \tag{60}$$

Sometimes it is useful to express a wrench in the form

$$w = \begin{bmatrix} \underline{f} \\ \underline{m}_0 \end{bmatrix} \tag{61}$$

where $\underline{f}$ is the resultant force and $\underline{m}_0$ is the resultant moment about the origin due to the system of forces and couples.

Plucker [1866] used the axis coordinates of a line (vector) to describe the instantaneous angular velocity of a rigid body about an axis which he called a rotary force in analogy with ordinary forces. Angular velocity may be expressed in line vector form by

$$\Omega = \begin{bmatrix} \underline{r} \times \underline{\Omega} \\ \underline{\Omega} \end{bmatrix} = \begin{bmatrix} \underline{\Omega}_0 \\ \underline{\Omega} \end{bmatrix} \tag{62}$$

where $\underline{\Omega}$ expresses the direction and magnitude of the angular velocity and where $\underline{r} \times \underline{\Omega}$ represents the rectilinear velocity of a point on the (extended) rigid body which is coincident with the origin.

A translational or rectilinear velocity of a rigid body may be effected as the sum of two successive distinct angular velocities with equal magnitudes and opposite orientations

$$V = \begin{bmatrix} \underline{r}_1 \times \underline{\Omega} \\ \underline{\Omega} \end{bmatrix} + \begin{bmatrix} \underline{r}_2 \times (-\underline{\Omega}) \\ -\underline{\Omega} \end{bmatrix} = \begin{bmatrix} (\underline{r}_1 - \underline{r}_2) \times \underline{\Omega} \\ \underline{0} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{V}_0 \\ \underline{0} \end{bmatrix} . \qquad (63)$$

It was first demonstrated by Charles (see Ball [1900]) that the most general displacement of a rigid body between two positions may be effected by a rotation about an axis together with a translation parallel to the axis. Ball called this motion a twist on a screw. Here a twist is always used to denote an instantaneous displacement since such motions are communitive in the sense that they may be successively applied to the body in any order. Analogous to (59), the motion resulting from successive angular velocities Ω_i and translational velocities V_i is equivalent to a twist on a screw

$$T = \Sigma\Omega_i + \Sigma V_i . \qquad (64)$$

For instantaneous motion, Charles' theorem may be stated as, the most general displacement of a rigid body is equivalent to an angular velocity about an axis together with a translational velocity parallel to the axis and

$$T = \begin{bmatrix} \underline{r} \times \underline{\Omega} + h\underline{\Omega} \\ \underline{\Omega} \end{bmatrix} = \begin{bmatrix} \underline{r} \times \underline{\Omega} \\ \underline{\Omega} \end{bmatrix} + \begin{bmatrix} h\underline{\Omega} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{T}_0 \\ \underline{T} \end{bmatrix} . \qquad (65)$$

It is sometimes useful to express twist (65) in the form

$$T = \begin{bmatrix} \underline{V}_0 \\ \underline{\Omega} \end{bmatrix} \tag{66}$$

where $\underline{\Omega}$ is the angular velocity associated with the twist and $\underline{V}_0$ is the translational velocity of a point on the (extended) rigid body which is coincident with the origin.

The compositions of wrenches w_i and twists T_i are given by the analogous relations

$$w = \Sigma\ w_i \tag{67}$$

$$T = \Sigma\ T_i \tag{68}$$

which have somewhat different interpretations. In (67) it is implied that the wrenches w_i are acting concurrently or in parallel, for example, on a single rigid body. However in (68), the twists T_i must act in succession or in a serial manner on the body. Thus the complementary concepts of parallel and serial composition are closely related to wrenches and twists which are themselves dual elements of mechanics.

Finally, there is a principle for wrenches and twists which is entirely analogous to the projective property of incidence for lines and reciprocity for screws, viz. the principle of virtual work. For example, assume that the instantaneous velocity of a rigid body is given by the twist T and the body is subjected to a wrench w. The instantaneous virtual work, or in other words the virtual power, may be expressed using (61) and (66) by

$$T^T w = \begin{bmatrix} \underline{V}_0 \\ \underline{\Omega} \end{bmatrix}^T \begin{bmatrix} \underline{f} \\ \underline{m} \end{bmatrix} = \underline{f} \cdot \underline{V}_0 + \underline{\Omega} \cdot \underline{m}_0 . \tag{69}$$

When the virtual work vanishes the twist and wrench are said to be reciprocal, a relationship which forms the basis for most of Ball's [1900] work. When a wrench is reciprocal to a twist, it is often referred to as a constraint wrench. It should be noted that in terms of Lagrangian mechanics, the components of T may be referred to as generalized velocities and the components of w may be referred to as generalized forces.

CHAPTER 4

INVERSIVE GEOMETRY

> The anxious precision of modern mathematics is necessary for accuracy, . . . it is necessary for research. It makes for clearness of thought and for fertility in trying new combinations of ideas. When the initial statements are vague and slipshod, at every subsequent stage of thought, common sense has to step in to limit applications and to explain meanings. Now in creative thought common sense is a bad master. Its sole criterion for judgment is that the new ideas shall look like the old ones, in other words it can only act by suppressing originality.
> A.N. Whitehead [1911, p. 157]

In this chapter a number of new mappings are introduced which have a direct connection with the elliptic polarity of screws in Euclidean space. These mappings were uncovered while investigating the significance of "orthogonality" applied to screws. Section 4.1 details the motivation for entailing this line of research and is explained using the concepts and notation developed in Chapter 2 and Chapter 3. "Orthogonality" of screws is currently being utilized for the hybrid control of robotic manipulators where it is apparently not recognized that the relation is noninvariant. "Orthogonality" and reciprocity of screws are closely related by the elliptic polarity.

The basic mappings of this chapter are derived in Section 4.2. The mappings involve the pitch and perpendicular vector of a screw onto an ordered quadruple. The mapping is not one-to-one since many screws map onto the same quadruple. However, the quadruple itself is then associated with a quaternion in a one-to-one mapping. A remarkable result is derived which shows that the elliptic polarity of screws induces the transformation of an associated quaternion into its inverse. A further interesting result is that as an operator, the associated quaternion transforms the direction and magnitude of a screw vector into that of its elliptic polar. The quadruple is also interpreted as a point in an inversive four-space. The elliptic polarity in Euclidean space is shown to induce, in the four-space, an inversion of the associated point through the unit hypersphere followed by a reflection in a hyperplane. A third mapping is established between the quadruples and radial pencils and bundles of screws with constant pitch. By way of the one-to-one and onto mappings of the quadruple with quaternions, points in an inversive four-space and radial pencils and bundles of screws, a three-way isomorphism is introduced. Further, an interesting relation is developed for the torsions and curvature vectors of a pair of circular helicies associated with two screws which are elliptic polars. Lastly in this section, the biquaternion in Euclidean space is determined which transforms a screw into its elliptic polar.

In Section 4.3 a new general derivation of Ball's circular representation of the two-system is presented. It is then used to describe the representation of the five special two-systems described by Hunt [1978]. Further, the previously introduced mappings are applied to Ball's planar representation from which it is given a new and more general characterization. It is shown to be an inversive plane where three of the special two-systems map to straight lines which may be considered as circles that pass through the singular point at infinity. Points on the inversive plane are also generalized to correspond to radial pencils of screws and circles to pencils of cylindroids. The quaternion mapping becomes a complex number representation of points which is used to model the inversive plane as a sphere via a stereographic projection.

The invariant properties of the elliptic polarity in Euclidean space are examined in Section 4.4. So-called self-polar screws are deduced in two manners, by using the previous mappings and by solving the eigenvalue problem associated with the elliptic polarity. Next, self-polar two-systems are studied via the planar mapping and several special cases are detailed. The complex number representation is also analytically applied to determine self-polar two-systems with what initially appears to be an erroneous result. However, by using the generalized interpretation of the inversive plane this discrepancy is resolved. Finally, a planar two-system is used to describe the effect of an origin translation on the elliptic polar of a screw.

Section 4.1 The Elliptic Polarity of Screws

> The importance of a result is largely relative, is judged differently by different men, and changes with the times and circumstances. It has often happened that great importance has been attached to a problem merely on account of the difficulties which it presented; and indeed if for its solution it has been necessary to invent new methods, noteworthy artifices, etc., the science has gained more perhaps through these than through the final result. In general we may call important all investigations relating to things which in themselves are important; all those which have a large degree of generality, or which unite under a single point of view subjects apparently distinct, simplifying and ellucidating them; all those which lead to results that promise to be the source of numerous consequences; etc. Corrado Segre [1891, p. 44]

In the following sections, several new mappings of screws are presented, central to which is a quaternion representation. The mappings were developed from an examination of the elliptic polarity of screws in Euclidean space. However, some of the mappings were observed previous to understanding their relationship with the elliptic polarity and resulted from an initial endeavor to explain the meaning of "orthogonality" in relationship to screws. It is perhaps useful to describe the motivating circumstances that resulted in the investigation and exposition of the material presented in this dissertation.

In the field of modern spatial kinematics and mechanisms, it has been a widespread practice to represent screw vectors in terms of dual vectors which employ Clifford's [1873] polar operator ω, where $\omega^2 = 0$. In the English language, early references in this field include, Denavit [1958], Yang and

Freudenstein [1964] and Dimentberg [1965] as well as the mathematical text by Brand [1947].

Screw vectors expressed in either ray coordinates or axis coordinates take on the same form when they are expressed as dual vectors. If only the group of rigid motions in Euclidean space is considered, their transformations are also the same. Historically, free vectors have been principally conceptualized as the difference of two points, Hamilton [1866]. Since a line vector may be considered as a vector which is restricted to slide along a given line, it is natural to interpret a line vector in terms of any two points on the line which are separated by a constant directed distance. Consequently, the ray coordinates of a line are an obvious conceptual expension of free vectors and the ray coordinates of a screw vector follows accordingly.

In the modern kinematics and mechanisms literature, two early introductions of ray coordinates may be found in Woo and Freudenstein [1970], and in Hunt [1970]. More extensive treatments are given by Hunt [1978] and Bottema and Roth [1979]. It should be noted however that Hunt [1978, p. 309] calculates the coordinates of a line formed by the intersection of two planes. He then comments on the apparent duality between this formulation and the expression of line coordinates using two points. Unfortunately, this avenue of investigation was not pursued. Recently, Lipkin and Duffy [1984a, 1984b] have explicitly reintroduced axis coordinates to this branch of the literature following the original conventions for twists and wrenches used by Plucker [1866]. This preliminary development is more fully detailed here in Chapters 2 and 3. Axis coordinates have also been used recently in an implicit manner by Mason [1978] and Salisbury [1982] where the geometric meaning of the coordinates is not recognized. Apparently they are introduced for the purpose of algebraic manipulations and simplifications.

Using a ray coordinate approach, Duffy and Sugimoto [1982] investigated further applications of screws to mechanisms. Previously, it was well known that the reciprocal product of a twist t and a wrench w, both expressed in ray coordinates, may be given by

$$t^T \tilde{\Delta} w = 0. \qquad (1)$$

This equation may represent the state of a body in static equilibrium which is acted upon by an impressed wrench w while being free to twist about t, that is, the virtual work vanishes. However, it was not then recognized that $\tilde{\Delta}$ represented the identical correlation of lines and screws (see Section 2.2). Nevertheless, it was apparent that (1) could be expressed in a form similar to

$$T^T w = 0 \qquad (2)$$

where

$$T = \tilde{\Delta} t. \qquad (3)$$

Since T and w satisfied a "standard" inner product which vanished in (2), T and w were said to be "orthogonal" screws. Further, it was also said that (3) represented a one-to-one correspondence between screws t which are reciprocal to w and screws T which are "orthogonal" to w.

The misinterpretation of (2) is not unique, but essentially may be found throughout most of the literature concerning hybrid control theory commencing with Mason [1978] and is detailed in a subsequent section. With the notation used here, the problem becomes obvious. Equation (2) does not represent what is referred to as a "standard" inner product which conveys a meaning of "orthogonality," but is instead a relationship akin to incidence which has been detailed in Section 2.2. Analogously,

$$W^T x = 0 \qquad (4)$$

relates that the point x and the plane W are incident and thus "orthogonality" is an inappropriate characterization of this projective relation.

Equation (3) represents the transformation of a screw in ray coordinates into the same screw in axis coordinates and thus its physical interpretation remains invariant while its representation is altered. Therefore, (2) is merely another rerpresentation of the static equilibrium condition (1). However, at the commencement of the investigation presented here, it was not then recognized that (3) was a transformation of dual coordinates. Instead, the result of (3) was interpreted in terms of ray coordinates and thus the transformation was viewed as a type of collineation

$$t' = \hat{\Delta} t. \tag{5}$$

In Section 3.2, (5) has been identified as the elliptic polarity of screws expressed as a collineation and is tabulated in Table 3.2.1.

It is very important to note, that without a thorough geometrical development from basic principles, it is extremely difficult to clearly delineate projective from metrical relations. This is particularly evident by inspection of Table 3.2.1 where similar appearing expressions are placed along sides of each other. Correct interpretation of these relations is greatly facilitated by a distinctive notation which enables discrimination of correlations from collineations and ray coordinates from axis coordinates.

"Orthogonality" of screws expressed in ray coordinates has an appearance similar to (2),

$$t'^{T} w = 0 \tag{6}$$

and may be also expressed in the form

$$t'^{T} \tilde{I}_6 w = t'^{T} \tilde{\Gamma}_1 w = 0. \tag{7}$$

From Section 3.2, it is evident that (7) represents the condition that t' and w are elliptic conjugates or in other words are elliptic-orthogonal. Nevertheless, prior to recognizing this, "orthogonality" was examined in Euclidean space although

Woo and Freudenstein [1970] had already noted that it was not an invariant relationship. However, by first considering the elliptic polarity of screws in Euclidean space (5), a number of rather interesting mappings were uncovered. Further, it led to a generalization of Ball's [1900] circular representation of the two-system of screws and a new theory of hybrid control. These developments are detailed in subsequent sections.

The remainder of this section establishes results for the elliptic polarity of screws in Euclidean space which are used as a foundation for developments in the sequel. Since it is now clear that it is in fact the elliptic polarity which is being analyzed, there is no loss in generality by using ray coordinates exclusively. The development is directly analogous for axis coordinates and no new relationships are apparent by employing them. The following analysis applies to screw vectors which for brevity are simply referred to as screws.

First of all, in order to interpret the elliptic polarity in Euclidean space, it is necessary that the point which is coincident with the origin remains fixed. That is, only rotations and reflections through the origin are to be admitted. These transformations form a subgroup of Euclidean collineations as well as a subgroup of elliptic collineations.

This may be demonstrated by considering a screw p and its elliptic polar p' which is given by

$$p' = \hat{\Delta} p. \tag{8}$$

Since $\hat{\Delta}$ is an involution matrix, the inverse relation is similar

$$p = \hat{\Delta}p' \tag{9}$$

where

$$\hat{\Delta}\hat{\Delta} = \hat{I}_6 \quad \text{and} \quad \hat{\Delta} = \hat{\Delta}^T. \tag{10}$$

For a collineation acting on p,

$$q = \hat{K}p \tag{11}$$

it is desired that the collineation transforms p' in the same manner

$$q' = \hat{K}p' \tag{12}$$

where q' denotes the elliptic polar of q and

$$q' = \hat{\Delta}q. \tag{13}$$

Substituting (11), (12) in (13) yields

$$p' = (\hat{K}^{-1}\ \hat{\Delta}\ \hat{K})p. \tag{14}$$

Since p may be any screw, then comparing (8) with (14) gives the relation

$$\hat{K}^{-1}\ \hat{\Delta}\ \hat{K} = \hat{\Delta} \tag{15}$$

which may also be expressed by

$$\hat{\Delta}\hat{K} = \hat{K}\hat{\Delta}. \tag{16}$$

The necessity that $\hat{\Delta}$ and $\hat{K}$ commute in (16) is an obvious consequence of (11) and (12) for which the collineation is required to transform p and p' in the same manner. Multiplying (16) by the elliptic polarity in correlation form, $\tilde{I}_6$, yields

$$\hat{K}^{-1}\ \tilde{\Delta}\ \hat{K} = \tilde{\Delta} \qquad (17)$$

where clearly $\tilde{I}_6$ and $\hat{K}$ commute. In Section 3.3 it was shown that Euclidean collineations satisfy the tetrahedronal relationship in the form

$$\hat{K}^T\ \tilde{\Delta}\ \hat{K} = \tilde{\Delta}. \qquad (18)$$

Since $\hat{K}$ must simultaneously satisfy both (17) and (18), this results in

$$\hat{K}^T = \hat{K}^{-1} \qquad (19)$$

which implies that $\hat{K}$ must also be an elliptic collineation.

The only Euclidean collineations which satisfy (19) are rotations $\hat{K}_e$ and a reflection through the origin $\hat{K}_\sigma$, both of which are tabulated in Table 3.3.2. Under either collineation, the origin remains fixed. Translations do not have this property and thus must be excluded from further consideration of allowable collineations.

In the following development, it is useful to introduce the vector form for screws given in Section 3.4,

$$p = \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} \quad , \quad p' = \begin{bmatrix} \underline{p}' \\ \underline{p}_0' \end{bmatrix} \qquad (20)$$

for which (8) and (9) become

$$p' = \hat{\Delta} p = \begin{bmatrix} \underline{p}_0 \\ \underline{p} \end{bmatrix} , \qquad p = \hat{\Delta} p' = \begin{bmatrix} \underline{p}_0' \\ \underline{p}' \end{bmatrix} \tag{21}$$

and where

$$\underline{p}_0 = \underline{r} \times \underline{p} + h\underline{p} , \quad \underline{p}_0' = \underline{r}' \times \underline{p}' + h'\underline{p}' . \tag{22}$$

In (22), the parameters h, h' respectively denote the pitches of p, p'. The quantities $\underline{r}, \underline{r}'$ most generally denote the vectors from the origin of any points on the respective axes of p, p'. Here however, $\underline{r}, \underline{r}'$ will be consistently used to denote the normal vectors to the respective axes whereas in Section 3.4 the notation $\underline{r}_n$ was employed for this purpose.

From (22), the pitch and normal vectors for p are given by

$$h = \frac{\underline{p} \cdot \underline{p}_0}{\underline{p} \cdot \underline{p}} , \quad \underline{r} = \frac{\underline{p} \times \underline{p}_0}{\underline{p} \cdot \underline{p}} \tag{23}$$

and for p' by

$$h' = \frac{\underline{p}' \cdot \underline{p}_0'}{\underline{p}' \cdot \underline{p}'} , \quad \underline{r}' = \frac{\underline{p}' \times \underline{p}_0'}{\underline{p}' \cdot \underline{p}'} . \tag{24}$$

Substituting (21) in (23) and using (24) results in

$$h = \frac{h'}{h'^2 + \underline{r}'^2} , \quad \underline{r} = \frac{-\underline{r}'}{h'^2 + \underline{r}'^2} . \tag{25}$$

Conversely, substituting (21) in (24) and using (23) yields an analogous relationship

$$h' = \frac{h}{h^2 + \underline{r}^2} \quad , \quad \underline{r}' = \frac{-\underline{r}}{h^2 + \underline{r}^2} \tag{26}$$

where the notation $\underline{r}^2 = \underline{r} \cdot \underline{r}$ is employed. Therefore from (25) and (26), the pitch and normal vector corresponding to a screw p or p', can be expressed directly in terms of these quantities corresponding to its elliptic polar screw, p' or p respectively. The transformations (25) and (26) are most generally referred to as Cremona and quadratic birational transformations, see Snyder, et al. [1970], for example.

From (25), (26) it is clear that the vectors $\underline{r}$ and $\underline{r}'$ are antiparallel and thus the common normal between p and p' intersects the origin as illustrated in Fig. 4.1.1. A unit vector for this normal line pointing in the direction of $\underline{r}$ is given by

$$\underline{e} = \frac{\underline{r}}{\|\underline{r}\|} \ . \tag{27}$$

The angle θ between the directed axes of the screws is measured by a right-hand rotation of $\underline{p}$ into $\underline{p}'$ about $\underline{e}$ and is specified by the relations

$$\cos\theta = \frac{\underline{p}}{\|\underline{p}\|} \cdot \frac{\underline{p}'}{\|\underline{p}'\|} = \frac{\underline{p}}{\|\underline{p}\|} \cdot \frac{\underline{p}_0}{\|\underline{p}_0\|}$$

$$= \frac{h}{(h^2 + \underline{r}^2)^{\frac{1}{2}}} = \frac{h'}{(h'^2 + \underline{r}'^2)^{\frac{1}{2}}} \tag{28}$$

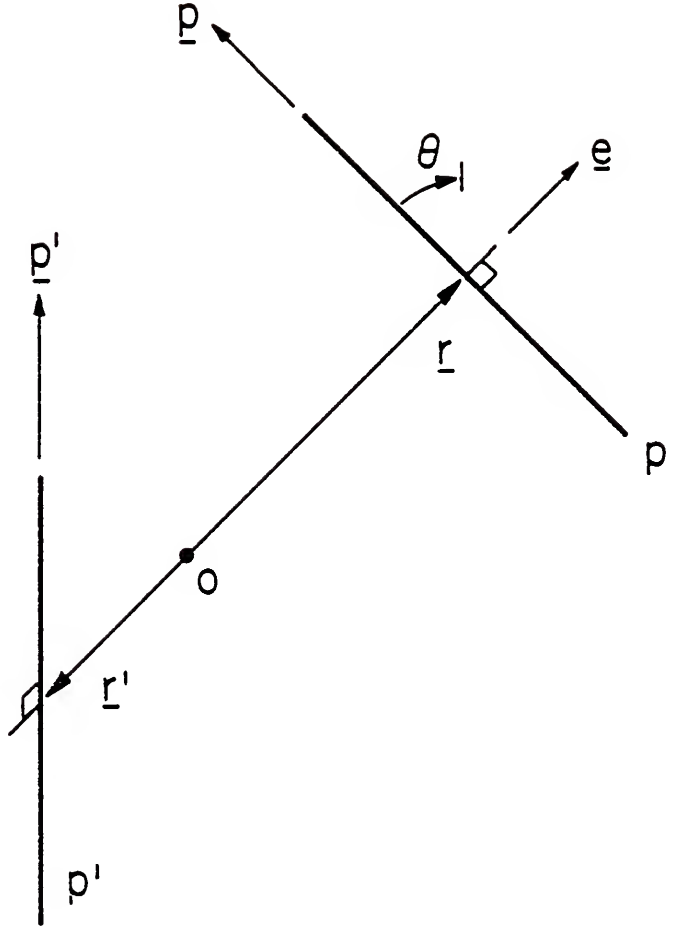

Figure 4.1.1 A screw p and its elliptic polar p'.

$$\sin\theta = \frac{\underline{p}}{\|\underline{p}\|} \times \frac{\underline{p}'}{\|\underline{p}'\|} \cdot \underline{e} = \frac{\underline{p}}{\|\underline{p}\|} \times \frac{\underline{p}_0}{\|\underline{p}_0\|} \cdot \underline{e}$$

$$= \frac{\underline{r} \cdot e}{(h^2 + \underline{r}^2)^{\frac{1}{2}}} = \frac{-\underline{r}' \cdot \underline{e}}{(h'^2 + \underline{r}'^2)^{\frac{1}{2}}} \qquad (29)$$

from which it follows that

$$\tan\theta = \frac{\underline{r} \cdot \underline{e}}{h} = \frac{-\underline{r}' \cdot \underline{e}}{h'} . \qquad (30)$$

Relations (25), (26) and (28), (29) are used frequently in the following sections.

Section 4.2 Quaternion Mappings

> It appears, then, from the foregoing discussion, that for the complete determination, of what we have called the geometrical Quotient of two co-initial Vectors, a System of Four Elements, admitting each separately of numerical expression, is generally required. Of these four elements, one serves to determine the relative length of the two lines compared; and the other three are in general necessary, in order to determine fully their relative direction. Again, of these three latter elements, one represents the mutual inclination, or elongation, of the two lines; or the magnitude (or quantity) of the angle between them; while the two others serve to determine the direction of the axis, perpendicular to their common plane, round which a rotation through that angle is to be performed, in a sense previously selected as the positive one (or towards a fixed and previously selected hand), for the purpose of passing (in the simplest way, and therefore in the plane of the two lines) from the direction of the divisor-line, to the direction of the dividend-line. And no more than four numerical elements are necessary, for our present purpose; because the relative length of two lines is not changed, when their two lengths are altered proportionally, nor is their relative direction changed, when the angle which they form is merely turned about, in its own plane. On account,

> then, of this essential connexion of that complex relation between two lines, which is compounded of a relation of lengths, and of a relation of directions, and to which we have given (by an extension from the theory of scalars) the name of a geometrical quotient, with a System of Four numerical Elements, we have already a motive for saying, that "the Quotient of two Vectors is generally a Quaternion."
> W.R. Hamilton [1866, pp. 112-113]

In this section, a number of new mappings of screws are presented where the elliptic polarity assumes a principal role. The most important mapping is one onto the quaternions where the elliptic polarity induces a transformation of a quaternion into its inverse.

The basic relation considered here is the mapping of a screw p onto the quadruple $(h,\underline{r})$, where h is the pitch and $\underline{r}$ is the normal vector from an origin (see Fig. 4.1.1) which is assumed throughout to be fixed. This mapping is clearly onto since for every quadruple $(h,\underline{r})$ there exists a screw which is its preimage. However, the mapping is not one-to-one since each quadruple is the image of numerous screws which each have the same pitch and normal vector. This correspondence is further detailed subsequently in this section.

It is very useful to associate the quadruple $(h,\underline{r})$ with a quaternion q where

$$q = h + \underline{r}. \qquad (1)$$

Analogously, the elliptic polar screw p' is mapped onto the quadruple $(h',\underline{r}')$ which is then associated with the quaternion of q' where

$$q' = h' + \underline{r}'. \tag{2}$$

It should be noted that (1) is a convenient notation for the usual expression of a quaternion in terms of its components

$$q = h + r_1 i + r_2 j + r_3 k \tag{3}$$

where the complex numbers i, j, k have the multiplicative properties

$$i^2 = j^2 = k^2 = ijk = -1 \tag{4}$$

and

$$i = jk = -kj,\ j = ki = -ik,\ k = ij = -ji. \tag{5}$$

Briefly, quaternions form a communitive group under addition and a noncommunitive group under multiplication which, together with the distributive property of multiplication over addition, characterizes it as a skew-field or equivalently as a division ring, see Birkhoff and MacLane [1965], for example. The multiplication of two quaternions may be conveniently expressed using vector notation

$$\begin{aligned} q_1 q_2 &= (h_1 + \underline{r}_1)(h_2 + \underline{r}_2) \\ &= (h_1 + h_2 - \underline{r}_1 \cdot \underline{r}_2) + (h_1\, \underline{r}_2 + h_2\, \underline{r}_1 + \underline{r}_1 \times \underline{r}_2). \end{aligned} \tag{6}$$

The quaternion inverses of q and q' are given by

$$q^{-1} = \frac{h}{h^2 + \underline{r}^2} - \frac{\underline{r}}{h^2 + \underline{r}^2} \tag{7}$$

$$q'^{-1} = \frac{h'}{h'^2 + \underline{r}'^2} - \frac{\underline{r}'}{h'^2 + \underline{r}'^2} . \tag{8}$$

Using (4.1.26) in (7) and (4.1.25) in (8) yields

$$q^{-1} = h' + \underline{r}' \quad , \quad q'^{-1} = h + \underline{r} \tag{9}$$

for which it is clear from (4.1.25), (4.1.26) that

$$q^{-1} = q' \quad , \quad q'^{-1} = q. \tag{10}$$

Therefore from (10), the transformation of a screw p into its elliptic polar p' induces in the quaternion mapping, a transformation of q into its inverse. This is a rather remarkable connection between the quaternion mapping and the elliptic polarity of screws.

Another interesting property of the mapping is that as an operator, the quaternion q transforms the vector $\underline{p}$ into $\underline{p}'$ by quaternion multiplication (6) and

$$\underline{p}' = q\underline{p} \quad , \quad \underline{p}' = q'\underline{p}. \tag{11}$$

This transformation is detailed by

$$\begin{aligned} q\underline{p} &= (h + \underline{r})\underline{p} = h\underline{p} - \underline{r} \cdot \underline{p} + \underline{r} \times \underline{p} \\ &= h\underline{p} + \underline{r} \times \underline{p} = \underline{p}_0 = \underline{p}' \end{aligned} \tag{12}$$

and using (10) to obtain the inverse relation. The transformation may also be developed by expressing q and q' as scalar multiples of unit quaternions,

$$q = (h^2 + \underline{r}^2)^{\frac{1}{2}} (\cos\theta + \underline{e}\sin\theta) \tag{13}$$

$$q' = (h'^2 + \underline{r}'^2)^{\frac{1}{2}} (\cos\theta - \underline{e}\sin\theta) \tag{14}$$

where $\underline{e}$ and θ are defined by (4.1.27) - (4.1.29) and where the scalar factors are the respective quaternion norms. Since the direction $\underline{e}$ is mutually perpendicular to both $\underline{p}$ and $\underline{p}'$ (see Fig. 4.1.1), then by (13), q transforms $\underline{p}$ into $\underline{p}'$ by a rotation θ about $\underline{e}$ together with a dilation $(h^2 + \underline{r}^2)$ which is also given by the ratio

$$\frac{\|\underline{p}'\|}{\|\underline{p}\|} = \frac{\|\underline{p}_0\|}{\|\underline{p}\|} = (h^2 + \underline{r}^2)^{\frac{1}{2}}. \tag{15}$$

It is also useful to consider the quadruple $(h,\underline{r})$ as the nonhomogeneous coordinates of a point in a four-dimensional inversive space or equivalently a conformal space, see Coxeter [1969], Guggenheimer [1963]. An inversive space may be considered almost identical to a Euclidean space with the exception that the hyperplane at infinity is replaced by a single point. Briefly, there are two sets of invariant properties which are applied in inversive geometry. One set consists of the Euclidean metrics for distance and angle which, for four-dimensional and n-dimensional spaces, is detailed by Sommerville [1929]. For two-dimensional and three-dimensional spaces, a thorough discussion of the invariant properties for inversive transformations is given by Coolidge [1916]. The important relationship between the two is that inversive transformations preserve angles but not distances. Thus in inversive geometry it is typical to refer to purely Euclidean (or sometimes metric) properties and purely inversive properties, a distinction which becomes lost when the situation at hand involves only angles. This co-existence of Euclidean and inversive properties is discussed further by Graustein [1930]. In two dimensions, useful models of inversive planes are given by stereographic projections of a sphere from a pole onto either an equatorial plane, see Klein [1884], or onto a plane tangent at the opposite pole, see Klein [1908].

The properties of the elliptic polarity screws which are induced onto the inversive space may be deduced by first squaring and adding the transformation formulas in either

$$h = \frac{h'}{h'^2 + \underline{r}'^2} \quad , \quad \underline{r} = \frac{-\underline{r}'}{h'^2 + \underline{r}'^2} \qquad (4.1.25)$$

or in

$$h' = \frac{h}{h^2 + \underline{r}^2} \quad , \quad \underline{r}' = \frac{-\underline{r}}{h^2 + \underline{r}^2} \qquad (4.1.26)$$

which yields

$$(h^2 + \underline{r}^2)(h'^2 + \underline{r}'^2) = 1. \qquad (16)$$

This is the fundamental formula which defines a general inversion in the unit hypersphere

$$h^2 + \underline{r}^2 = 1. \qquad (17)$$

The inversive transformation (4.1.26), may be represented as the composition of three distinct transformations, an "ordinary" inversion in the origin, which is also called a transformation of reciprocal radii by Klein [1908], a reflection through the origin, and a reflection in a hyperplane. The "ordinary" inversion in the origin has the form

$$h_1 = \frac{h}{h^2 + \underline{r}^2} \quad , \quad \underline{r}_1 = \frac{\underline{r}}{h^2 + \underline{r}^2} \; . \tag{18}$$

The reflection through the origin is given by

$$h_2 = -h_1 \quad , \quad \underline{r}_2 = -\underline{r}_1 \tag{19}$$

and the reflection in the hyperplane h = 0 yields

$$h' = -h_2 \quad , \quad \underline{r}' = \underline{r}_2 . \tag{20}$$

In the inversive space, the reflection through the origin is also induced by a reflection in the origin in the preimage space. That is, for the screw p, its reflection is given by the collineation $\hat{K}_\sigma$ in Table 3.3.2 with $\sigma_3 = -I_3$ and

$$\hat{K}_\sigma \; p = \begin{bmatrix} -I_3 & \cdot \\ \cdot & I_3 \end{bmatrix} p = \begin{bmatrix} -\underline{p} \\ \underline{p}_0 \end{bmatrix} . \tag{21}$$

Substituting (21) in the formulas for h and $\underline{r}$ in (4.1.23) yields the induced transformation,

$$\begin{bmatrix} -I_1 & \cdot \\ \cdot & -I_3 \end{bmatrix} \begin{bmatrix} h \\ \underline{r} \end{bmatrix} = \begin{bmatrix} -h \\ -\underline{r} \end{bmatrix}. \qquad (22)$$

Analogously, for the rotation $\hat{K}_e$ listed in Table 3.3.2,

$$K_e\ p = \begin{bmatrix} e & \cdot \\ \cdot & e \end{bmatrix} p = \begin{bmatrix} e\ \underline{p} \\ e\ \underline{p}_0 \end{bmatrix} \qquad (23)$$

and the corresponding induced transformation is given by

$$\begin{bmatrix} I_1 & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} h \\ \underline{r} \end{bmatrix} = \begin{bmatrix} h \\ e\ \underline{r} \end{bmatrix}. \qquad (24)$$

It should be noted that the hyperplane h = 0 in the inversive space, with the exception of the point at infinity, is identical to the finite region of three-dimensional Euclidean space. This is because the vector $\underline{r}$ corresponds to the point on the screw axis which is closest to the origin. All such points comprise finite Euclidean space as well as the hyperplane h = 0 with the singular point deleted. This is exactly analogous to the quaternion mapping where q becomes a pure vector for h = 0.

The inversion specified by (4.1.26) is a one-to-one mapping for all points in the inversive space. The point at the origin $(0, \underline{0})$ is transformed into the point at infinity, which is represented here symbolically as $(\infty, \underline{\infty})$, and conversely $(\infty, \underline{\infty})$ is transformed into $(0, \underline{0})$.

In addition to the mappings of the quadruple $(h, \underline{r})$ onto quaternions and points in an inversive space, a third one-to-one mapping may be established between (h, $\underline{r}$) and sets of screws in Euclidean space. In general, the quadruple (h, $\underline{r}$) may be associated with what is referred to here as a radial pencil of screws with pitch h. Referring to Fig. 4.1.1, the radial pencil, or simply the pencil, may be generated by rotating a screw p about its normal axis through the origin which is indicated by the vector $\underline{r}$. To each screw in the pencil (h, $\underline{r}$) there corresponds an elliptic polar screw p' which forms a second pencil (h', $\underline{r}'$).

The first singularity in this representation occurs for $\underline{r} = \underline{0}$ and the quadruple (h, $\underline{0}$) corresponds to a bundle of screws through the origin with pitch h. For $h \neq 0$, the elliptic polars of this bundle also form a bundle through the origin but with pitch

$$h' = \frac{1}{h} . \tag{25}$$

For h = 0, the quadruple (0, $\underline{0}$) corresponds to a bundle of lines through the origin. The elliptic polars of this bundle, form a bundle on the plane at infinity which corresponds to the second singularity in the mapping. That is, the bundle at infinity is analogous to the singular point at infinity (∞, $\underline{\infty}$) in the inversive four-space.

For the particular case where h = 0, $\underline{r} \neq 0$, the quadruple (0, $\underline{r}$) corresponds to a radial pencil of lines and the elliptic polars correspond to a second radial pencil of lines (0, $\underline{r}'$) which are both illustrated in Fig. 4.2.1. From (4.1.25) and

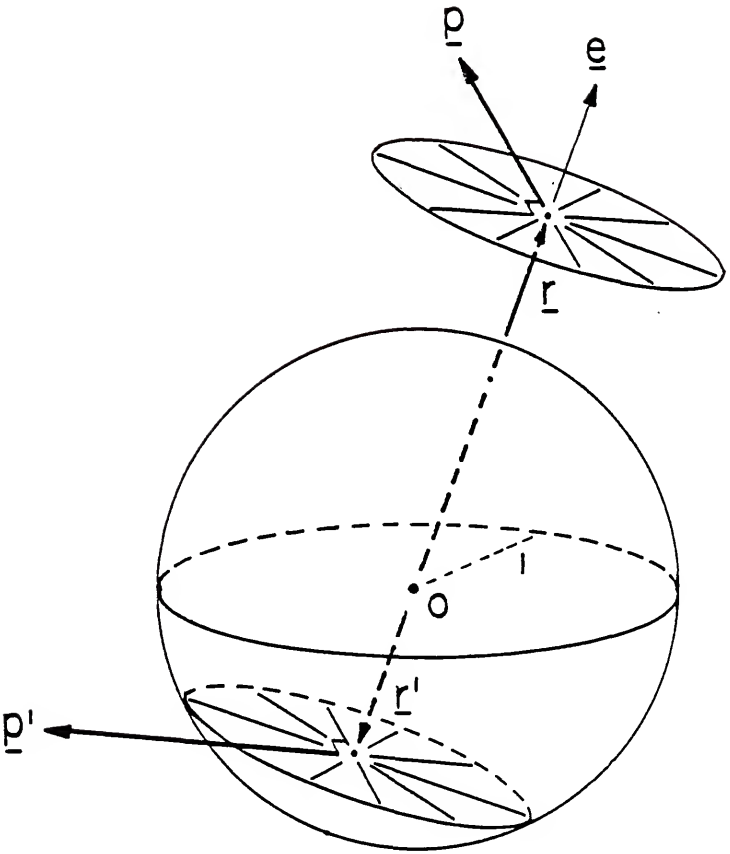

Figure 4.2.1 A radial pencil of lines and its elliptic polar.

(4.1.26), the vectors $\underline{r}$ and $\underline{r}'$ are related by

$$\underline{r} = \frac{-\underline{r}'}{\underline{r}'^2} \quad , \quad \underline{r}' = \frac{-\underline{r}}{\underline{r}^2} \; . \tag{26}$$

Squaring either relation in (26) yields

$$(\underline{r}^2)(\underline{r}'^2) = 1 \tag{27}$$

which is the fundamental formula for an inversion through the unit sphere

$$\underline{r} \cdot \underline{r} = 1. \tag{28}$$

Using the cosine and sine relations (4.1.28), (4.1.29), the angle between corresponding lines p and p' in each pencil is $\theta = \pi/2$. Therefore, the elliptic polarity induces a transformation such that the point at the center of a radial pencil is inverted through the unit sphere and each line of the pencil is turned through a right angle. Lines which are tangent to the sphere are clearly transformed into lines which are tangent at the corresponding antipodal points.

For the three mappings of the quadruple (h, $\underline{r}$) as quaternions, points in an inversive four-space and radial pencils (and bundles) of screws in Euclidean space, it is possible to establish a three-way isomorphism. Firstly, it is necessary to delete (∞, $\underline{\infty}$) from consideration in the isomorphism since it has no correspondence in relation to quaternions. Thus, the singular point at infinity in the inversive space and the bundle of lines at infinity in Euclidean space must be excluded, leaving only finite elements for all three mappings. Further, since there is no quaternion multiplicative inverse for $(0, \underline{0})$, then the inversion of the origin in inversive space must be left undefined as well as the elliptic polarity of the bundle of lines through the origin in Euclidean space.

Therefore, with these restrictions in place, the properties of the algebra of quaternions, namely addition, multiplication and their inverses together with the vector space properties of an algebra, can be interpreted for points in the inversive space and for radial pencils (and bundles) of screws in Euclidean space via the one-to-one and onto mappings which induces the three-way isomorphism. The most interesting property is the correspondence between the quaternion multiplicative inversive, the inversion of a point through the unit hypersphere followed by its reflection in a hyperplane and the elliptic polarity of a radial pencil (or bundle) of screws in Euclidean space.

It is possible to remove some of the restrictions associated with $(\infty, \underline{\infty})$ in the isomorphisms by extending quaternions to include $(\infty, \underline{\infty})$ without necessarily giving it an interpretation. This extension would only be defined for the quaternion inverse and must be excluded from the properties of addition and multiplication. Analogous restrictions would also apply for the other two mappings.

There is a fourth mapping of the quadruple $(h, \underline{r})$ which can be related to the geometry of a circular helix. Figure 4.2.2 illustrates a portion of a circular helix which passes through the origin 0. The curve may be generated by a point on a rigid body that is twisting about a screw p with a pitch h at a constant angular velocity $\underline{p}$. At time $t = 0$, the point is initially coincident with the origin and the normal vector to the screw axis is given by $\underline{r}_0$. At any time t, the vector from the origin to the moving point is given by the vector loop equation

$$\underline{x} = \underline{r}_0 + h\,t\,\underline{p} - \underline{r} \tag{29}$$

where $\underline{r}$ is the normal vector to the axis from the moving point. The first three derivatives of (29) are given by

$$\dot{\underline{x}} = h\,\underline{p} - \underline{p} \times \underline{r} = \underline{p}_0 \tag{30}$$

$$\ddot{\underline{x}} = \underline{p} \times \underline{p}_0 \tag{31}$$

$$\dddot{\underline{x}} = \underline{p} \times (\underline{p} \times \underline{p}_0) . \tag{32}$$

The torsion and curvature vector of a curve are given by (see Coxeter [1969], for example),

$$\tau = \frac{\dot{\underline{x}} \times \ddot{\underline{x}} \cdot \dddot{\underline{x}}}{(\dot{\underline{x}} \times \ddot{x}) \cdot (\dot{\underline{x}} \times \ddot{\underline{x}})} \tag{33}$$

$$\kappa = \frac{\ddot{\underline{x}}}{\dot{\underline{x}} \cdot \dot{\underline{x}}} . \tag{34}$$

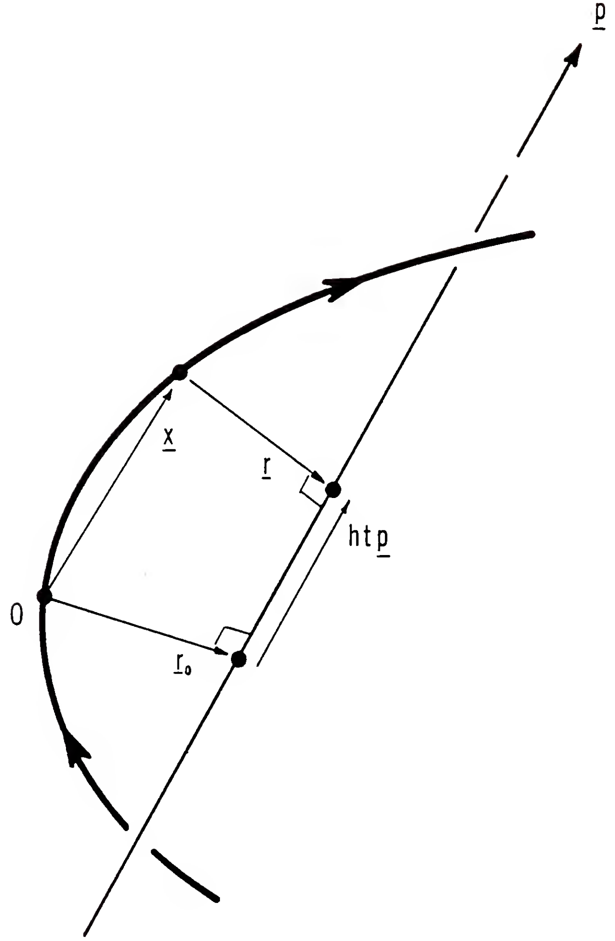

Figure 4.2.2 A helix through the origin associated with p.

Substituting (30) - (32) in (33), (34) yields after some manipulation

$$\tau = \frac{\underline{p}_0 \cdot \underline{p}}{\underline{p}_0 \cdot \underline{p}_0} = \frac{h}{h^2 + \underline{r}^2} \qquad (35)$$

$$\underline{\kappa} = \frac{\underline{p} \times \underline{p}_0}{\underline{p}_0 \times \underline{p}_0} = \frac{\underline{r}}{h^2 + \underline{r}^2} \qquad (36)$$

which may in turn be solved for h and $\underline{r}$ to give the symmetrical expressions

$$h = \frac{\tau}{\tau^2 + \underline{\kappa}^2} \quad , \quad \underline{r} = \frac{\underline{\kappa}}{\tau^2 + \underline{\kappa}^2} \qquad (37)$$

Further, the curvature of the helix is given by

$$\kappa = || \underline{\kappa} || = \frac{|| \underline{r} ||}{h^2 + \underline{r}^2} \qquad (38)$$

and since both the curvature and torsion of the curve are constant they fully describe its intrinsic properties. Curvature and torsion are also known as the first and second curvature and what is often referred to as third or total curvature is given by,

$$(\tau^2 + \underline{\kappa}^2)^{\frac{1}{2}} = (h^2 + \underline{r}^2)^{-\frac{1}{2}} \; . \qquad (39)$$

Analogously, for the elliptic polar screw p', there exists a corresponding helix through the origin whose torsion and curvature vector has the form

$$\tau' = \frac{h'}{h'^2 + \underline{r}'^2} \quad , \quad \underline{\kappa}' = \frac{\underline{r}'}{h'^2 + \underline{r}'^2} \tag{40}$$

which leads to the symmetrical expressions

$$h' = \frac{\tau'}{\tau'^2 + \underline{\kappa}'^2} \quad , \quad \underline{r}' = \frac{\underline{\kappa}}{\tau'^2 + \underline{\kappa}'^2} \tag{41}$$

The corresponding relations for the first and third curvatures are given by

$$\kappa' = \| \kappa' \| = \frac{\| r' \|}{h'^2 + \underline{r}'^2} \tag{42}$$

$$(\tau'^2 + \underline{\kappa}'^2)^{\frac{1}{2}} = (h'^2 + \underline{r}'^2)^{-\frac{1}{2}} \; . \tag{43}$$

Figure 4.2.3 illustrates two circular helices through the origin which are depicted on cylinders with radii $\| \underline{r} \|$ and $\| \underline{r}' \|$. Respectively, the helices correspond to the screws p, p' with pitches h, h' whose axes are in the directions of $\underline{p}$, $\underline{p}'$ and are inclined at the angle θ given by (4.1.28) and (4.1.29). From (35) - (37), (41), (42) it is possible to deduce the following symmetrical relations at the origin for which $\underline{r} = \underline{r}_0$,

$$(h, \underline{r}) = (\tau', -\underline{\kappa}') \tag{44}$$

$$(h', \underline{r}') = (\tau, -\underline{\kappa}) . \tag{45}$$

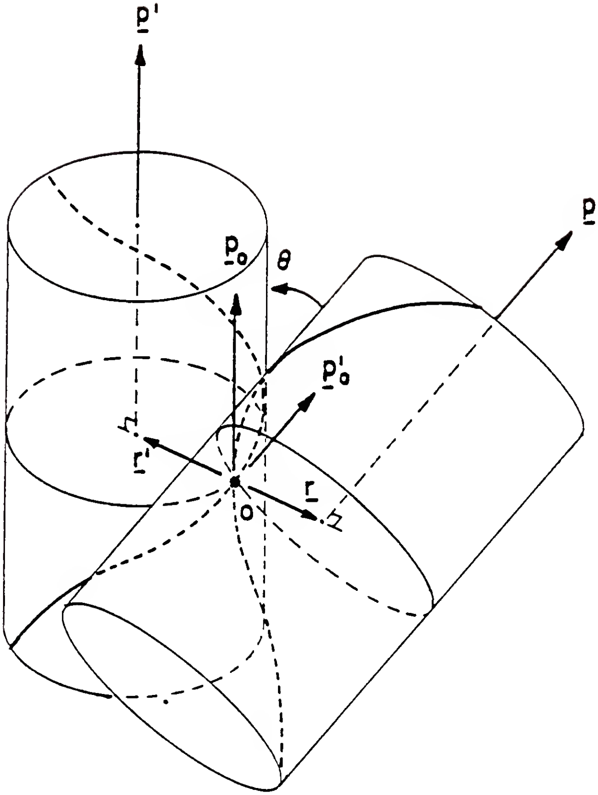

Figure 4.2.3 Two helices through the origin corresponding to a pair of screws which are elliptic polars.

Thus, the elliptic polarity transforms the quadruple (h, $\underline{r}$) into the torsion and the negative curvature vector through the origin (τ, $-\underline{\kappa}$) of the helix through the origin associated with its screw p. An analogous situation holds for p'. This is a rather remarkable relation which connects the elliptic polarity with the intrinsic properties (actually τ, $\| \underline{\kappa} \|$) of helices associated with screws.

Similar to the mapping of screws, all helices through the origin may be mapped onto the quadruples (h, $\underline{r}$). However, unlike radial pencils (and bundles) of screws, there are problems in trying to make the mapping one-to-one for pencils (and bundles) of such helices in the singular cases. As an example, consider the case for (h, $\underline{0}$) where the axis of screw p passes through the origin. The helix becomes a straight line for which both the curvature and torsion vanish although from (35),

$$\tau = \frac{1}{h} \tag{46}$$

which is inconsistent.

The helix mapping is not further investigated since the purpose here is not necessarily to determine another quaternion isomorphism. It is purpose enough to detail the interesting connections in which the quadruple (h, $\underline{r}$) is related to the torsion and negative curvature vector (τ, $-\underline{\kappa}$) through the elliptic polarity of screws.

Finally, it is also interesting to determine the biquaternion which transforms a screw p into its elliptic polar screw p'. Using Clifford's [1873] operator ω, where $\omega^2 = 0$, p and p' are expressed as the respective dual vectors

$$\bar{p} = \underline{p} + \omega\, \underline{p}_0 \tag{47}$$

$$\bar{p}' = \underline{p}' + \omega\, \underline{p}_0'. \tag{48}$$

The biquaternion to be determined is given by

$$\bar{q} = q + \omega\, q_0 \tag{49}$$

where q and q_0 are two quaternions. The transformation is specified by

$$\begin{aligned} \bar{p}' &= \bar{q}\, \bar{p} = (q + \omega\, q_0)(\underline{p} + \omega\, \underline{p}_0) \\ &= q\, \underline{p} + \omega(q\, \underline{p}_0 + q_0\, \underline{p}). \end{aligned} \tag{50}$$

Equating the real and dual parts of (48) and (50) yields

$$\underline{p}' = q\, \underline{p} \tag{51}$$

$$\underline{p} = q\, \underline{p}_0 - q_0\, \underline{p}. \tag{52}$$

Equation (51) is identical to the first equation in (11) and thus the expression for q is the same as defined by (1). Using $\underline{p}_0 = \underline{p}'$ in (52) and then substituting the result in (51) gives

$$(q^2 + q_0 - 1)\underline{p} = 0 \tag{53}$$

which for $\underline{p} \neq 0$ results in

$$q_0 = 1 - q^2. \tag{54}$$

The desired result is given by

$$\bar{q} = q + \omega(1 - q^2) \tag{55}$$

which may be expressed more symmetrically by

$$\begin{aligned} \bar{q} &= q[1 - \omega(q - q^{-1})] \\ &= q[1 - \omega(q - q')]. \end{aligned} \tag{56}$$

The inverse relation is simply,

$$\begin{aligned} \bar{q}' = \bar{q}^{-1} &= q^{-1} [1 + \omega(q - q^{-1})] \\ &= q' [1 + \omega(q - q')]. \end{aligned} \tag{57}$$

Equation (55) or (56) is rather interesting since it demonstrates that the same amount of information, namely (h, $\underline{r}$), is required to transform the screw p into p$'$ as is required to transform the vector $\underline{p}$ into $\underline{p}'$.

Section 4.3 Ball's Planar Representation of the Two-System

Song of the Screw.

A moving form or rigid mass,
Under whate'er conditions
Along successive screws must pass
Between each two positions.
It turns around and slides along-
This is the burden of my song.

The pitch of screw, if multiplied
By angle or rotation,
Will give the distance it must glide
In motion of translation.
Infinite pitch means pure translation,
And zero pitch means pure rotation.

Two motions on two given screws,
With amplitudes at pleasure,
Into a third screw-motion fuse,
Whose amplitude we measure
By parallelogram construction
(A very obvious deduction).

Its axis cuts the nodal line
Which to both screws is normal,
And generates a form divine,
Whose name, in language formal,
Is "surface-ruled of third degree."
Cylindroid is the name for me.

Rotation round a given line
Is like a force along,
If to say couple you decline,
You're clearly in the wrong;-
'Tis obvious, upon reflection,
A line is not a mere direction.

So couples with translations too
In all respects agree;
And thus there centres in the screw
A wondrous harmony
Of Kinematics and of Statics,-
The sweetest thing in mathematics.

The forces on one given screw,
With motion on a second,
In general some work will do,
Whose magnitude is reckoned
By angle, force, and what we call
The coefficient virtual.

Rotation now to force convert,
   And force into rotation;
Unchanged the work, we can assert,
   In spite of transformation.
And if two screws no work can claim,
Reciprocal will be their name.

Five numbers will a screw define,
   A screwing motion, six;
For four will give the axial line,
   One more the pitch will fix;
And hence we always can contrive
One screw reciprocal to five.

Screws-two, three, four or five, combined
   (No question here of six),
Yield other screws which are confined
   Within one screw complex.
Thus we obtain the clearest notion
Of freedom and constraint of motion.

In complex III, three several screws
   At every point you find,
Or if you one direction choose,
   One screw is to your mind;
And complexes of order III.
Their own reciprocals may be.

In IV, wherever you arrive,
   You find of screws a cone,
On every line of complex V.
   There is precisely one;
At each point of this complex rich,
A plane of screws have given pitch.

But time would fail me to discourse
   Of Order and Degree;
Of Impulse, Energy and Force,
   And Reciprocity.
All these and more, for motions small,
Have been discussed by Dr. Ball.

-Anonymous, in Moritz [1914, pp. 320-322].

Ball [1900] has demonstrated how the screws of a two-system may be represented on a plane. In this section, a new and more general derivation of the planar representation is presented in which the mapping of the two-system is deduced from two general screws. Then using this derivation, the mappings of the five special two-systems described by Hunt [1978] are detailed.

Further, the three-way quaternion isomorphism derived in Section 4.2, is applied to the planar mapping. This results in a new interpretation of Ball's planar representation which is presented on an inversive plane using complex numbers in which the elliptic polarity induces a conformal mapping. The analysis further generalizes Ball's results and yields important insight in the interpretation of equations relating to the planar mapping.

As in the previous section, there is no loss in generality by using ray coordinates exclusively since the relations for axis coordinates are identically analogous. Also, the mappings presented here are applicable for screw vectors which for brevity are referred to as screws.

Consider an axis which passes through an origin 0 and the totality of screws which intersect the axis at right angles. Each screw may be mapped onto the pair (h, r) where h is the pitch of the screw and r is the directed distance from the origin to the screw. The pair (h, r) may be used to denote a point on a plane in the mapping space.

A two-system may be defined as all screws which are linearly dependent on two distinct screws. It is well known, see Ball [1900], Hunt [1978], that the screw axes of a two-system in general form the generators of a ruled surface which is known as the cylindroid. As illustrated in Fig. 4.3.1, the generators of the cylindroid are at right angles

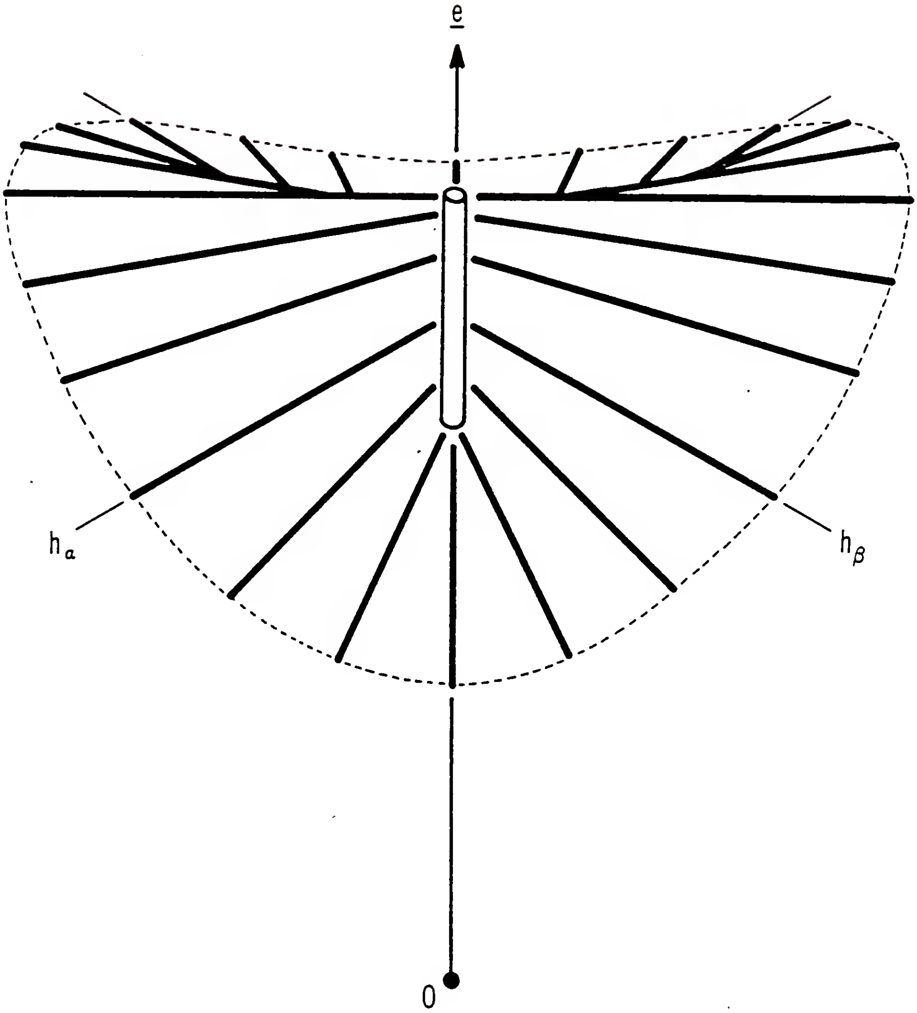

Figure 4.3.1 The cylindroid.

to the nodal axis on which the origin is located. Ball has shown that the screws associated with a cylindroid map onto a circle in the (h, r) plane and here a more general derivation is presented.

Any screw p in the two-system may be expressed in the form

$$p = \lambda_1 p_1 + \lambda_2 p_2 \tag{1}$$

where p_1, p_2 are any distinct screws in the two-system and λ_1, λ_2 are arbitrary scalar multipliers which are not simultaneously zero. Each screw may be expressed in a vector form

$$p_i = \begin{bmatrix} \underline{p}_i \\ \underline{p}_{0i} \end{bmatrix} = \begin{bmatrix} \underline{p}_i \\ \underline{r}_i \times \underline{p}_i + h_i \underline{p}_i \end{bmatrix}, \quad i = 1,2 \tag{2}$$

where h_i, represents the pitch and $\underline{r}_i$ represents a vector from the origin corresponding to any point on the screw axis for which the direction ratios are given by $\underline{p}_i$.

Referring to Fig. 4.3.1, the pitch and directed normal distance from the origin for a screw p which lies on the cylindroid of the associated two-system is given by

$$h = \frac{\underline{p} \cdot \underline{p}_0}{\underline{p} \cdot \underline{p}} \tag{3}$$

$$r = \underline{r} \cdot \underline{e} = \frac{\underline{p} \times \underline{p}_0}{\underline{p} \cdot \underline{p}} \cdot \underline{e} \tag{4}$$

where $\underline{e}$ is a unit vector along the nodal axis. (It is noted that it is not essential to place the origin on the nodal axis, but doing so simplifies the discussion.) Substituting (2) in (1) and the result in both (3) and (4) respectively yields, after some manipulation, a pair of quadratic equations

$$\lambda^2(h-h_1) + \lambda(2h-h_1-h_2)c_{12} - (r_1-r_2)s_{12} + (h_1-h_2) = 0 \tag{5}$$

$$\lambda^2(r-r_1) + \lambda(2r-r_1-r_2)c_{12} - (h_1-h_2)s_{12} + (r_1-r_2) = 0 \tag{6}$$

where the parameter λ is defined by the ratio

$$\lambda = \frac{\lambda_1 \| \underline{p}_1 \|}{\lambda_2 \| \underline{p}_2 \|} \tag{7}$$

and where the angle α_{12} between p_1 and p_2 is defined by

$$c_{12} = \cos \alpha_{12} = \frac{\underline{p}_1 \cdot \underline{p}_2}{\|\underline{p}_1\| \cdot \|\underline{p}_2\|} \tag{8}$$

$$s_{12} = \sin \alpha_{12} = \frac{\underline{p}_1 \times \underline{p}_2 \cdot \underline{e}}{\|\underline{p}_1\| \cdot \|\underline{p}_2\|} \ . \tag{9}$$

Equations (5) and (6) may be expressed more symmetrically by

$$\lambda^2(h_1^*) + \lambda(h_1^*+h_2^*)c_{12} + (r_1^*-r_2^*)s_{12} + (h_2^*) = 0 \tag{10}$$

$$\lambda^2(r_1^*) + \lambda(r_1^*+r_2^*)c_{12} + (h_1^*-h_2^*)s_{12} + (r_2^*) = 0 \tag{11}$$

where

$$h_i^* = h-h_i \quad , \quad r_i^* = r-r_i . \tag{12}$$

For two general quadratic equations,

$$a_i\lambda^2 + b_i\lambda + c_i = 0 \ , \quad i = i,2 \tag{13}$$

the necessary and sufficient condition that the two equations have a common root is given by the eliminant which may be expressed in the form

$$|ab| \cdot |bc| - |ac|^2 = 0 \tag{14}$$

where $|ab| = a_1b_2 - a_2b_1$, etc. Using (14) to eliminate the parameter λ from (10) and (11) yields, after some rearranging,

$$[(h_1^* - h_2^*)^2 + (r_1^* - r_2^*)^2] \cdot \begin{bmatrix} h_1^* \\ r_1^* \end{bmatrix}^T \begin{bmatrix} s_{12} & -c_{12} \\ c_{12} & s_{12} \end{bmatrix} \begin{bmatrix} h_2^* \\ r_2^* \end{bmatrix} = 0. \tag{15}$$

It should be noted from (12) that the first factor in (15) is a scalar whereas the second factor is a quadratic form in h and r. The explicit circular representation of the cylindroid may be obtained by substituting (12) into the quadratic part of (15) which yields

$$(h - \tfrac{1}{2}[(h_1+h_2) + k_{12}(r_1-r_2)])^2 + (r-\tfrac{1}{2}[(r_1+r_2) - k_{12}(h_1-h_2)])^2$$

$$= \tfrac{1}{4}(1+k_{12}^2)[(h_1-h_2)^2 + (r_1-r_2)^2] \tag{16}$$

where

$$k_{12} = \frac{c_{12}}{s_{12}} = \cot \alpha_{12} \; . \tag{17}$$

The circle equation (16) is the most general representation of a nondegenerate cylindroid on the (h, r) plane.

There is an interesting interpretation on the (h, r) plane of the quadratic form in (15). It may also be expressed as the bilinear form given by

$$\underline{v}_2^T \, A \, \underline{v}_1 = 0 \tag{18}$$

where

$$\underline{v}_i = - \begin{bmatrix} h_i^* \\ r_i^* \end{bmatrix} = \begin{bmatrix} h_i - h \\ r_i - r \end{bmatrix} , \; i = 1,2 \tag{19}$$

and

$$A = \begin{bmatrix} \cos(\alpha_{12} + \pi/2) & -\sin(\alpha_{12} + \pi/2) \\ \sin(\alpha_{12} + \pi/2) & \cos(\alpha_{12} + \pi/2) \end{bmatrix} . \tag{20}$$

Figure 4.3.2 illustrates the mapping of the cylindroid on the (h, r) plane where $\underline{v}_1$, $\underline{v}_2$ are represented as two vectors from the variable point (h, r) on the circle to the points (h_1, r_1) and (h_2, r_2) corresponding to the given screws p_1, p_2. Using a result given by Ball [1900] (see also Hunt [1978]), the angle between $\underline{v}_1$ and $\underline{v}_2$ is given by α_{12}, which is the same angle between the screw axes of p_1, p_2. In (20),

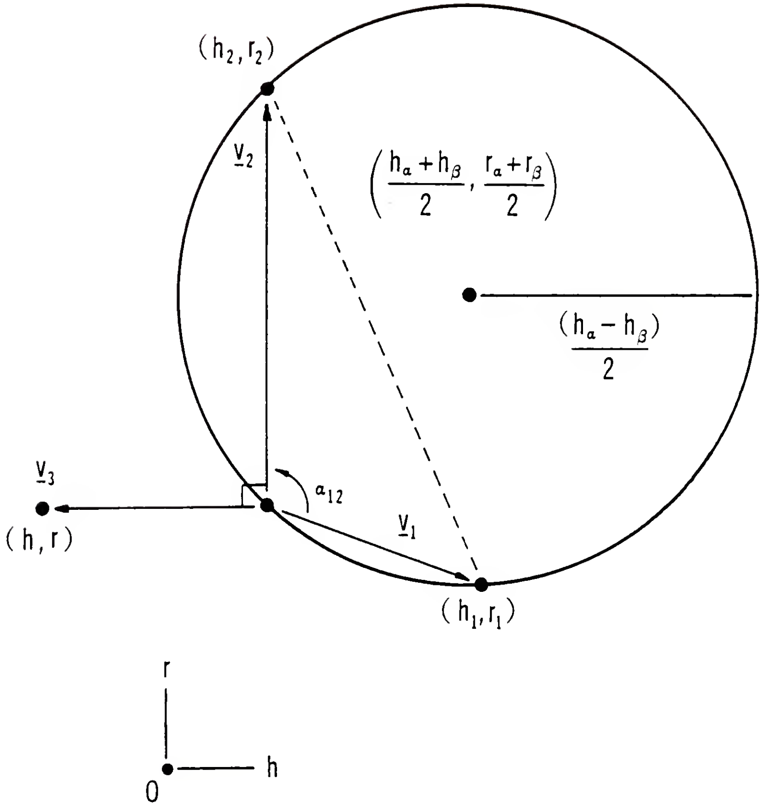
(h_2, r_2)
$\underline{v}_2$
$\left(\frac{h_\alpha + h_\beta}{2}, \frac{r_\alpha + r_\beta}{2}\right)$
$\frac{(h_\alpha - h_\beta)}{2}$
$\underline{v}_3$
(h, r)
α_{12}
$\underline{v}_1$
(h_1, r_1)
r
h
0

Figure 4.3.2 The circular representation of the cylindroid.

the matrix A represents a right-hand rotation through the angle $\alpha_{12} + \pi/2$. In the figure, the vector given by

$$\underline{v}_3 = A\ \underline{v}_1 \tag{21}$$

is clearly normal to $\underline{v}_2$ and thus

$$\underline{v}_2^T\ \underline{v}_3 = \underline{v}_2^T\ A\ \underline{v}_1 = 0 \tag{22}$$

which is equivalent to (28) and thus to the quadratic form in (15) using (19). The scalar factor in (15) may be expressed using (12) by

$$(h_1 - h_2)^2 + (r_1 - r_2)^2 \tag{23}$$

which represents the length of the chord between the two points. This scalar vanishes only if the two screws p_1, p_2 map to the same point.

It is possible to reduce the general equation (16) to a simpler and more familiar form by employing the principal screws of the two-system. Briefly, the principal screws are a pair of reciprocal screws which intersect at a right angle at the midpoint of the cylindroid. They have the maximum and minimum pitches h_α and h_β and are illustrated in Fig. 4.3.1. Using the principal screws for p_1, p_2 and substituting

$$h_1 = h_\alpha,\ h_2 = h_\beta,\ r_\alpha = r_\beta,\ k_{12} = 0 \tag{24}$$

in (16) yields

$$[h-\tfrac{1}{2}(h_\alpha+h_\beta)]^2+[r-\tfrac{1}{2}(r_\alpha+r_\beta)]^2=[\tfrac{1}{2}(h_\alpha-h_\beta)]^2 \qquad (25)$$

which conveniently expresses the center and radius of the circle as shown in Fig. 4.3.2. Selecting the origin 0 at the midpoint of the cylindroid yields a result given by Hunt [1978],

$$[h-\tfrac{1}{2}(h_\alpha+h_\beta)]^2+r^2=[\tfrac{1}{2}(h_\alpha-h_\beta)]^2. \qquad (26)$$

Hunt [1978] describes five special two-systems where the cylindroid becomes degenerate. It is interesting to examine the mappings on the (h, r) plane for these special systems.

For the first special two-system, the pitches of the principal screws are equal and the cylindroid degenerates to a radial pencil of screws with pitch h_α and normal distance r_α. Equation (25) becomes

$$(h-h_\alpha)^2+(r-r_\alpha)^2=0 \qquad (27)$$

which is the equation of a point circle located at (h_α, r_α). Also, the scalar factor in (15) which corresponds to the chord in Fig. 4.3.2 vanishes.

For the remaining four cases, one or both of the principal screws has infinite pitch and it is not possible to use (16) or (25). When one principal screw has finite pitch, then it is possible to select two alternative finite screws which are then used to determine the mapping. For this situation let the infinite and finite principal screws be respectively, $p_1 = p_\alpha$ and $p_2 = p_\beta$. Then, any screw p in the two-system is given by

$$p = \lambda_1 \begin{bmatrix} \underline{0} \\ \underline{p}_{01} \end{bmatrix} + \lambda_2 \begin{bmatrix} \underline{p}_2 \\ \underline{p}_{02} \end{bmatrix} = \begin{bmatrix} \lambda_2\, \underline{p}_2 \\ \lambda_1\, \underline{p}_{01} + \lambda_2\, \underline{p}_{02} \end{bmatrix} \tag{28}$$

from which it is clear that all the screw axes must be either parallel or antiparallel to the direction $\underline{p}_2$. Thus selecting an alternative pair of screws p_1, p_2 which are both finite, the angle between them is $\alpha_{12} = 0$ or π and by (17), $k_{12} = \infty$. Equation (16) is not useful since dividing through by k_{12}^2 and passing the limit results in an identical relation equivalent to 0 = 0. The reason for this is that the eliminant (14) expresses the condition which the two quadratics (10), (11) must satisfy in order that they have a common root or factor. If both equations have a common factor which is independent of a relation amongst the coefficients of the two quadratics, then the eliminant must vanish identically since the equations always have a common root.

For the case under consideration, $\alpha_{12} = 0, \pi$ and

$$c_{12} = \sigma_1 \quad , \quad s_{12} = 0 \tag{29}$$

where $\sigma_1 = \pm 1$. Substituting (29) in (10), (11) yields

$$\lambda^2 (h_1^*) + \lambda (h_1^* + h_2^*) \sigma_1 + h_2^* = 0 \tag{30}$$

$$\lambda^2(r_1^*) + \lambda(r_1^* + r_2^*)\sigma_1 + r_2^* = 0 \tag{31}$$

It may be easily verified that the eliminant of (30), (31) using (14) vanishes identically. Equations (30), (31) may be expressed in the form

$$\lambda^2 + \sigma_1(1 + \mu_h)\lambda + \mu_h = 0 \tag{32}$$

$$\lambda^2 + \sigma_1(1 + \mu_r)\lambda + \mu_r = 0 \tag{33}$$

where,

$$\mu_h = \frac{h_2^*}{h_1^*}, \quad \mu_r = \frac{r_2^*}{r_1^*}. \tag{34}$$

The two quadratics may be factored as

$$(\lambda + \sigma_1)(\lambda + \sigma_1 \mu_h) = 0 \tag{35}$$

$$(\lambda + \sigma_1)(\lambda + \sigma_1 \mu_r) = 0. \tag{36}$$

From (7), it may be verified that the value $\lambda = -\sigma_1$, corresponds to the infinite screw of the two-system. Thus, deleting the common factor from (35), (36) yields the two linear relations

$$\lambda + \sigma_1 \mu_h = 0 \tag{37}$$

$$\lambda + \sigma_1 \mu_r = 0 \tag{38}$$

from which λ is easily eliminated,

$$\begin{vmatrix} 1 & \sigma_1 \mu_h \\ 1 & \sigma_1 \mu_r \end{vmatrix} = 0. \qquad (39)$$

Using (34) and (12) in (39) yields the desired result which is expressed in determinant form by

$$\begin{vmatrix} h & (h_1-h_2) \\ r & (r_1-r_2) \end{vmatrix} + \begin{vmatrix} h_1 & h_2 \\ r_1 & r_2 \end{vmatrix} = 0. \qquad (40)$$

and clearly represents a straight line through the points (h_1, r_1) and (h_2, r_2). It is important to note that on an inversive or conformal plane, straight lines are in general equivalent to circles which pass through the singular point at infinity (∞, ∞). The remaining four cases of special two-systems are now summarized.

The second special two-system may be specified by the principal screws p_1, p_2 in (28) where

$$\underline{p}_2 \cdot \underline{p}_{01} = 0. \qquad (41)$$

It consists of parallel screws of constant pitch which lie on a plane. Each screw axis intersects at right angles the line through the origin given by

$$\begin{bmatrix} \underline{p}_{01} \times \underline{p}_{02} \\ \underline{0} \end{bmatrix} \qquad (42)$$

and is analogous to the nodal line of a nondegenerate cylindroid which is also specified by (42). Substituting $h_1 = h_2$ in (40) for two finite screws of the system yields the mapping on the (h, r) plane

$$h = h_1 \tag{43}$$

which is simply a line parallel to the r axis.

The principal screws of the third special two-system are both infinite and thus so are all its screws

$$p = \lambda_1 \begin{bmatrix} \underline{0} \\ \underline{p}_{01} \end{bmatrix} + \lambda_2 \begin{bmatrix} \underline{0} \\ \underline{p}_{02} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \lambda_1 \, \underline{p}_{01} + \lambda_2 \, \underline{p}_{01} \end{bmatrix} . \tag{44}$$

Since every screw has infinite pitch and directed distance, the two-system is mapped onto the singular point at infinity, (∞, ∞). By analogy with the first special two-system, it may be considered as a point circle at infinity.

The fourth special two-system is a generalization of the second special two system where the restriction (41) is deleted. The principal screws are given by (28) and the system is comprised of parallel screws on a plane. The line through the origin meeting each screw axis at right angles is also given by (42). The mapping on the (h, r) plane is given by the general straight line (40) and thus the pitch values vary as a linear function of r.

The fifth special two-system is given by the principal screws of (28) subject to,

$$\underline{p}_{01} \times \underline{p}_{02} = \underline{0}. \tag{45}$$

The two-system consists of all screws that lie on a single line plus an infinite one. Substituting $r_1 = r_2$, for two finite screws, in (40) yields the mapping on the (h, r) plane

$$r = r_1 \tag{46}$$

which is a line parallel to the h axis. The mappings of the special two-system are used subsequently in the following sections.

It is important to explain the role of the three-way quaternion isomorphism of Section 4.2 in connection with the plane representation of the two-system. Firstly, the quaternion mapping of a screw was expressed by

$$q = h + \underline{r} = h + r_1 i + r_2 j + r_3 k \ . \tag{47}$$

Aligning one of the coordinate axes in Euclidean space with the unit vector $\underline{e}$ in Fig. 4.3.1, then a screw of the two-system which is mapped to the pair (h, r) may be represented by a complex number

$$q = h + ir, \ (i^2 = 1). \tag{48}$$

This complex number represents a special quaternion in which two components are clearly always zero.

Similarly, the mapping of a screw onto a point in an inversive four-space

$$(h, \underline{r}) = (h\ r_1\ r_2\ r_3) \tag{49}$$

may for a screw of the two-system be mapped onto a point which lies on a plane (∞^2 points) formed by two of the four coordinate axes and represented by (h, r). This is exactly the inversive plane of Ball's representation which has been already discussed.

The one-to-one mapping of radial pencils (and bundles) of screws onto the quadruples $(h, \underline{r})$ has a very useful interpretation in connection with the two-system. Firstly, it is assumed that there exists a single axis through the origin whose direction is denoted by $\underline{e}$ as in Fig. 4.3.1. Every pencil of screws with pitch h that is centered on the axis may be mapped to the pair (h, r). Reciprocally, every pair may be mapped to such a pencil. In Section 4.2, the quadruple $(h, \underline{0})$ was a singular case which represented a bundle of screws through origin which pitch h. However, for the pair (h, 0) a singular case in the representation is avoided since a direction $\underline{e}$ has already been assumed for the axis carrying the pencils.

Thus the three-way quaternion isomorphism is now used as a three-way complex number isomorphism. That is, the pair (h, r) may represent either a complex number, a point on an inversive space or a radial pencil of screws. It is important to note how this isomorphism agrees with the previous analysis for the two-system. The first special two-system was shown to be a radial pencil of screws which pitch h. Its mapping on the (h, r) plane was a point circle which consisted of a single point, for real h and r. Thus it supports the general mapping of radial pencils as points on the inversion plane.

The mapping of two-systems on the (h, r) plane must also be interpreted in this manner. A nondegenerate cylindroid maps onto a unique circle, as does a single infinity of cylindroids which may be generated by rotating the cylindroid about its nodal axis. Thus a circle represents what is referred to here as a pencil of cylindroids. Analogously, this applies to the degenerate cylindroids which map to a line. It is very important to have this comprehensive viewpoint of the planar representation of the two-system. Otherwise, transformed equations in the (h, r) plane may appear erroneous. This is detailed for a specific example in the following section where a circle cannot be interpreted as a single cylindroid. It is noted that Ball [1900] considered a circle to represent a single cylindroid in which he assumed that the screw of maximum pitch always points in a constant direction. This is satisfactory for mapping many properties of two-system onto the (h, r) plane.

In Section 4.2 it was demonstrated that elliptic polarity of screws in Euclidean space induced the transformation of a quaternion into its inverse. Analogously, in the complex number representation this becomes

$$q = h + ir = \frac{h' - ir'}{h'^2 + r'^2} = q'^{-1} \tag{50}$$

$$q' = h' + ir' = \frac{h - ir}{h^2 + r^2} = q^{-1}. \tag{51}$$

On the (h, r) plane the elliptic polarity may be represented as an inversion the unit circle

$$q_1 = \frac{h + ir}{h^2 + r^2} \tag{52}$$

followed by a reflection through the origin

$$q_2 = -q_1 = -h_1 - ir_1 \tag{53}$$

and then followed by a reflection in the line h = 0,

$$q' = -h_2 + ir_2 \ . \tag{54}$$

It is useful to interpret the topology of an inversive plane by a stereographic projection of a unit sphere onto an equatorial plane as shown in Fig. 4.3.3. Following the development by Klein [1884], a spatial coordinate frame is located at the sphere center 0 with two of its axes coincident with the planar (h, r) coordinate frame. A line through the pole n and any point x on the sphere cuts the plane at the corresponding point q. As the distance between q and the origin increases in any direction, the corresponding point x approaches the pole n which is a useful visual device for the singular point at infinity (∞, ∞). The stereographic projection preserves inversive properties such as angle. A circle on the sphere is mapped to a circle unless

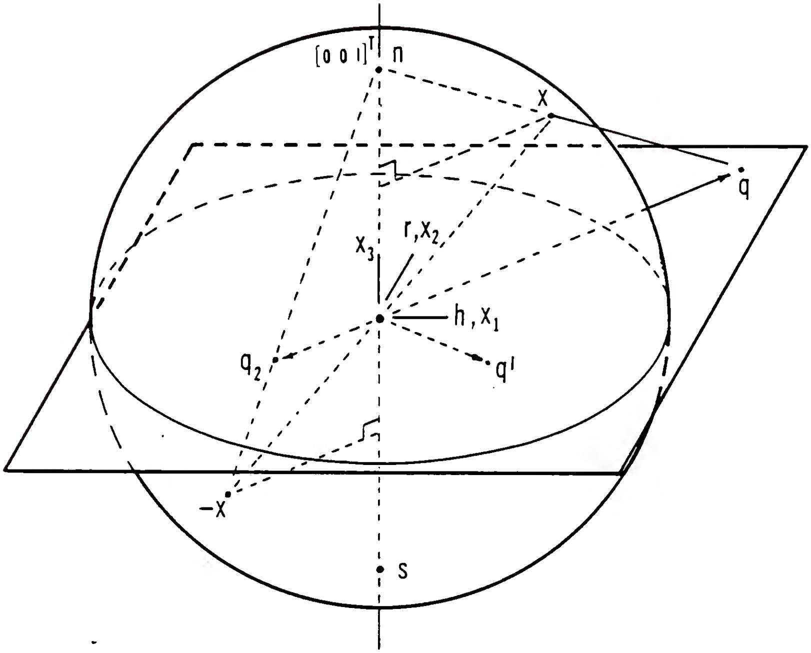

Figure 4.3.3 A stereographic projection modeling the inversive plane as the surface of a sphere.

it passes through the pole n in which it is mapped to a straight line, that is a circle through (∞, ∞).

Using similar triangles in Fig. 4.3.3, it is easily deduced that the coordinates of q may be expressed in terms of x by

$$q = h + ir = \frac{x_1 + ix_2}{1 - x_3} . \qquad (55)$$

The elliptic polar of q is given by

$$q^{-1} = q' = h' + ir' = \frac{x_1 - ix_2}{1 + x_3} . \qquad (56)$$

The point -x is antipodal to x and using similar triangles its projection on the plane has the form

$$q_2 = h_2 + ir_2 = \frac{-x_1 - ix_2}{1 + x_3} . \qquad (57)$$

This is the same point given in (53) which is formed by an inversion of q in the unit circle (52) followed by the reflection through the origin in (53). Thus reflecting q_2 in the line h = 0, (54), produces the elliptic polar q' as shown in Fig. 4.3.3. The stereographic projection of a sphere onto a plane is a useful device since the sphere serves as finite model of the inversive plane which is particularly valuable for interpreting properties relating to the singular point at infinity.

Section 4.4 Self-Polar Screw Systems

> What distinguishes the straight line and circle more than anything else, and properly separates them for the purpose of elementary geometry? Their self-similarity. Every inch of a straight line coincides with every other inch, and off a circle with every other off the same circle. Where, then, did Euclid fail? In not introducing the third curve, which has the same property--the screw. The right line, the circle, the screw--the representations of translation, rotation, and the two combined--ought to have been the instruments of geometry. With a screw we should never have heard of the impossibility of trisecting an angle, squaring the circle, etc. A. DeMorgan, in Graves [1889, Vol. 3, pp. 342]

In this section the elliptic polarity of screws in Euclidean space is examined by identifying the various invariant properties of the transformation. For the two-system of screws, this is facilitated by using the planar mapping introduced in Section 4.3. As previously demonstrated in Section 4.1, it must be assumed that the point which is coincident with the origin remains fixed since translations are not elliptic collineations and thus do not preserve the invariant properties of the elliptic polarity. However, Euclidean rotations about the origin forms a subgroup of elliptic collineations and thus preserve the invariant properties of the elliptic polarity.

As in the previous section, there is no loss in generality in using ray coordinates exclusively since the relations are analogous for axis coordinates. However, in this section it will be convenient to use the notation for screw vectors, although the investigation for the most part deals with screws themselves.

A screw which is equivalent to its elliptic polar is defined here as a self-polar screw. Two distinct methods are presented to deduce self-polar screws, the first of which uses the mapping of Section 4.1. A screw p is mapped onto the quadruple $(h, \underline{r})$ and the elliptic polar screw p' is mapped onto $(h', \underline{r}')$ where

$$h' = \frac{h}{h^2 + \underline{r}^2}, \quad r' = \frac{-\underline{r}}{h^2 + \underline{r}^2}. \tag{1}$$

For a screw to be self-polar it is necessary but not sufficient that

$$h' = h \quad \text{and} \quad \underline{r}' = \underline{r}. \tag{2}$$

Substituting (2) in (1) yields the conditions

$$h = \pm 1, \quad \underline{r} = 0. \tag{3}$$

Further from Section 4.1, the ratio of magnitudes for a screw p and its polar p' is given by

$$\frac{||p'||}{||\underline{p}||} = (h^2 + \underline{r}^2)^{\frac{1}{2}} \tag{4}$$

and the angle between the two axes is specified by

$$\cos\theta = \frac{h}{(h^2 + \underline{r}^2)}, \quad \sin\theta = \frac{\underline{r}\cdot\underline{e}}{(h^2 + \underline{r}^2)^{\frac{1}{2}}}. \tag{5}$$

In both cases the magnitude ratio (4) is unity. For $h = 1$, the angle from (5) is $\theta = 0$ and the screw vector is thus invariant. However, for the case $\theta = \pi$, the direction of the screw vector reverses and the screw vector is not invariant. However, the associated screw is invariant since its axis and pitch remain unaltered. Therefore, the necessary and sufficient conditions for a screw to be self-polar is that it is incident to the origin, $\underline{r} = 0$, and has pitch $h = \pm 1$. For a self-polar screw vector, the pitch is limited to $h = 1$.

The second method in which self-polar screws may be deduced is by solving the eigenvalue problem

$$p' = \lambda p \tag{6}$$

where λ is a scalar multiplier. Expressing the elliptic polarity in collineation form

$$p' = \hat{\Delta} p \tag{7}$$

and substituting in (6) yields

$$\lambda p = \hat{\Delta} p \tag{8}$$

or equivalently

$$\begin{bmatrix} -\lambda I_3 & I_3 \\ I_3 & -\lambda I_3 \end{bmatrix} p = 0. \tag{9}$$

Setting the determinant to zero, it may be easily transformed into

$$\begin{vmatrix} (1+\lambda)I_3 & \cdot \\ I_3 & (1-\lambda)I_3 \end{vmatrix} = 0 \qquad (10)$$

for which the characteristic equation is clearly

$$(1 + \lambda)^3 (1 - \lambda)^3 = 0 \qquad (11)$$

and thus there are two distinct eigenvalues $\lambda = \pm 1$, each of multiplicity three. From (8) it is readily deduced that λ is the magnitude ratio (4). Substitution of either eigenvalue in (9) yields a rank three matrix. The eigenscrews associated with $\lambda = \pm 1$ each form a bundle of screws through the origin with pitch $h = \pm 1$ respectively. Perhaps the most symmetrical set of eigenscrews may be obtained by choosing pairs of screws with pitches ±1 along three mutually perpendicular axes, the simplest case of which is along the coordinate axes and is specified by the columns of

$$\hat{\Lambda} = \begin{bmatrix} I_3 & I_3 \\ -I_3 & I_3 \end{bmatrix}. \qquad (12)$$

The screws of (12) are both coreciprocal and elliptic-orthogonal. As an elliptic collineation, $\hat{\Lambda}$ transforms Plucker coordinates into what have been called Study coordinates which have application in elliptic space, see

Coxeter [1965]. Thus this derivation of self-polar screws agrees exactly with the mapping method.

It is interesting to note that in both elliptic space and Euclidean space the screws which are left invariant under the elliptic polarity have pitches ±1. This is rather curious since the pitch of a screw in elliptic space is specified by a relation which is distinct from the one used in Euclidean space, as detailed in Section 3.2 and Section 3.3 respectively. The distinction between the two types of self-polar screws is that in Euclidean space their axes are incident to the origin whereas, in elliptic space they have no definite axes and thus they have the properties of free vectors according to Clifford [1873]. In contrast, screws in Euclidean space with infinite pitch (lines of infinity) have properties analogous to those of free vectors, see Section 3.4.

An n-system of screws, $1 \leq n \leq 6$, Ball [1900], is defined here as all linear combinations of n linearly independent screws p_i, $i = 1 \ldots n$ and

$$p = \lambda_1 p_1 + \lambda_2 p_2 + \ldots \lambda_n p_n \tag{13}$$

where the scalar multipliers λ_i are not all simultaneously zero and p is a screw of the n-system. Each screw p_i has an elliptic polar screw which may be used to form a basis for the corresponding elliptic polar n-system and

$$p' = \lambda_1 \hat{\Delta} p_1 + \lambda_2 \hat{\Delta} p_2 + \ldots \lambda_n \hat{\Delta} p_n . \tag{14}$$

An n-system which also contains the elliptic polar of every screw in the system is defined as a self-polar n-system. The previous results for a single screw n = 1 and its elliptic polar are now extended to the two-system of screws, n = 2.

A cylindroid corresponding to a self-polar two-system is defined as a self-polar cylindroid. A nondegenerate self-polar cylindroid may in general be constructed using a single screw which is not incident to the origin or on the plane at infinity, together with the elliptic polar of the screw

$$p = \lambda_1 p_1 + \lambda_2 p_1'. \tag{15}$$

Clearly, the elliptic polar of every screw is also in this system

$$p' = \lambda_2 p_1 + \lambda_1 p_1'. \tag{16}$$

Since p_1 and p_1' are not incident to the origin, they cannot be self-polar themselves and are thus linearly independent. As previously illustrated in Fig. 4.1.1, the line normal to both p_1 and p_1' passes through the origin and corresponds to the nodal axis of the cylindroid. The portion of the nodal axis which lies within the two extreme generators of the cylindroid is intersected at every point by two distinct screw axes, refer to Fig. 4.3.1 which is not however a self-polar cylindroid. For a nondegenerate cylindroid, the two screws through the origin may not be coaxial or have the same pitch. Thus by (3), the two screws through the origin of a nondegenerate self-polar cylindroid must themselves be self-polar screws, one with pitch +1 and the other with pitch -1.

Most generally, a self-polar cylindroid may be formed by selecting two linearly independent screws through the origin whose pitches are +1 or -1. If the two pitches are different and the screws are not coaxial, then this is the case of the nondegerate self-polar cylindroid previously discussed. If the two pitches are equivalent, then the cylindroid degenerates into a planar pencil of screws through the origin with either pitch +1 or -1 and every screw is self-polar. For example, using vector notation where only the ratios of coordinates are attended to, then for h_1, h_2 = +1, or h_1, h_2 = -1,

$$p = \lambda_1 \begin{bmatrix} \underline{p}_1 \\ \pm\underline{p}_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} \underline{p}_2 \\ \pm\underline{p}_2 \end{bmatrix} = \begin{bmatrix} (\lambda_1\underline{p}_1 + \lambda_2\underline{p}_2) \\ \pm(\lambda_1\underline{p}_1 + \lambda_2\underline{p}_2) \end{bmatrix} \tag{17}$$

and thus for every screw p, h = +1 or h = -1. In general, this self-polar system may be classified as a first special two-system which has been described in Section 4.3.

If the two self-polar screws through the origin have distinct pitches, one 1 and the other -1, then they may be coaxial. In this case the cylindroid degenerates to a single line through the origin which carries screws of all pitches and,

$$p = \lambda_1 \begin{bmatrix} \underline{p}_1 \\ \underline{p}_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} \underline{p}_1 \\ -\underline{p}_1 \end{bmatrix} = \begin{bmatrix} (\lambda_1+\lambda_2)\underline{p}_1 \\ (\lambda_1-\lambda_2)\underline{p}_1 \end{bmatrix} \qquad (18)$$

for which

$$h = \frac{\underline{p} \cdot \underline{p}_0}{\underline{p} \cdot \underline{p}} = \frac{(\lambda_1 - \lambda_2)}{(\lambda_1 + \lambda_2)} \ . \qquad (19)$$

Each screw p of pitch h has an elliptic polar p' in this two-system with $h' = 1/h$. This also includes a line at infinity for which $\lambda_1 = -\lambda_2$ and whose elliptic polar is the line given by $\lambda_1 = \lambda_2$. This self-polar system may be classified as a fifth special two-system as described in Section 4.3.

Thus for the single case of the nondegenerate self-polar cylindroid and the two cases of the degenerate self-polar cylindroids, it may be concluded that a two-system is self-polar, if and only if it contains two linearly independent screws through the origin with pitch ±1.

It is useful to examine self-polar two systems using the planar mapping presented in Section 4.3. A cylindroid whose nodal axis passes through the origin in Euclidean space is mapped onto a circle which may be expressed in the complex form

$$\| q - \gamma \|^2 = R^2 \qquad (20)$$

where

$$q = h + ir \quad , \quad \gamma = \alpha + i\beta \qquad (21)$$

or equivalently

$$(h - \alpha)^2 + (r - \beta)^2 = R^2. \tag{22}$$

On the (h, r) plane the elliptic polarity maps q into $q' = q^{-1}$ and thus the two-system is mapped onto a circle given by

$$\| q' - \gamma \| = R^2. \tag{23}$$

Using (1) and simplifying, (23) may be expressed by

$$\left(h - \frac{\alpha}{L}\right)^2 + \left(r + \frac{\beta}{L}\right)^2 = \frac{R^2}{L^2} \tag{24}$$

where

$$L = \alpha^2 + \beta^2 - R^2. \tag{25}$$

A necessary but not sufficient condition that a cylindroid be self-polar is that the circles (22) and (24) be identical. There are in general three distinguishable solutions:

$$R \neq 0,\ L = -1,\ h^2 + (r - \beta)^2 = \beta^2 + 1 \tag{26}$$

$$R = 0,\ L = 1,\ (h \pm 1)^2 + r^2 = 0 \tag{27}$$

$$R \neq 0,\ L = 1,\ (h - \alpha)^2 + r^2 = \alpha^2 - 1. \tag{28}$$

As illustrated by Fig. 4.4.1, (26) represents a single infinity of circles expressed in the variable parameter β. The circle centers are located at (0, β) and each passes through the two points (±1, 0) and thus have no other common

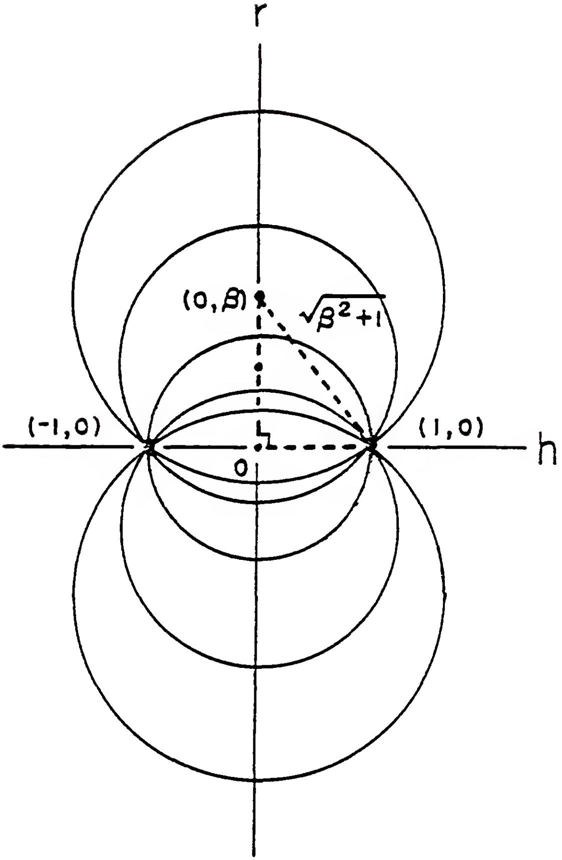

Figure 4.4.1 Mappings of self-polar two systems.

points of intersection. These points represent the self-polar screws of the cylindroid which have respective pitches ±1 and pass through the origin in Euclidean space as previously discussed. As $\beta \to \infty$ the corresponding circle flattens and in the limit becomes coincident with the h axis. This represents the second degenerate self-polar cylindroid that was discussed and forms a single line through the origin in Euclidean space. It is a fifth special two-system whose equation in the mapping is $r = 0$.

Equation (27) represents two point circles at $(\pm 1, 0)$. These correspond to the first degenerate self-polar cylindroid discussed which is a pencil of screws through the origin in Euclidean space with pitch +1 or -1 and was classified as a first special two-system.

As illustrated by Fig. 4.4.2, (28) represents a singular infinity of circles expressed in the variable parameter α. The circle centers are located at $(\alpha, 0)$ and intersect the unit circle at right angles but do not pass through the points $(\pm 1, 0)$. Here it is important to recall that a circle does not represent a single cylindroid, but rather a pencil of cylindroids obtained by a rotation about the nodal axis. It may be deduced that each cylindroid of the pencil is transformed by the elliptic polarity onto a second by rotating the first through a right angle about its nodal axis. This may be easily demonstrated using Fig. 4.4.2. The principal screws of the cylindroid have the maximum and minimum pitches and are respectively mapped onto the points on the h axis

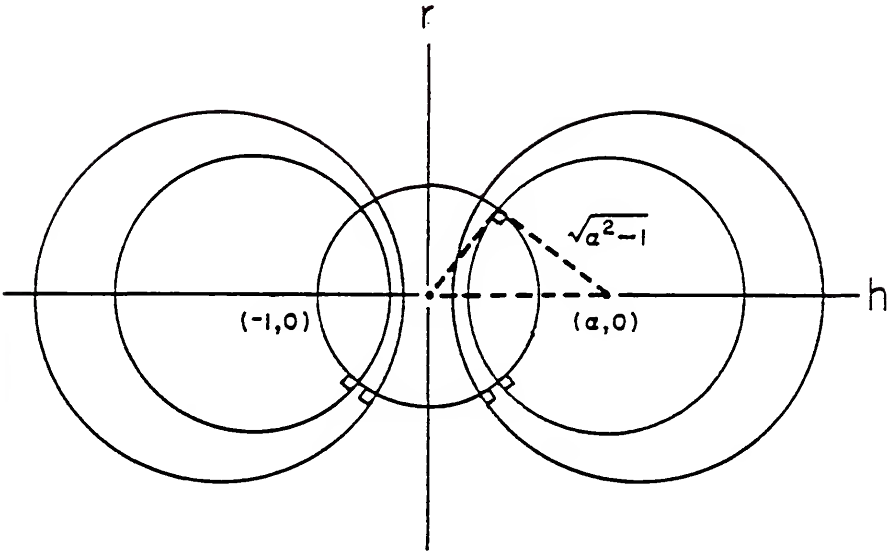

Figure 4.4.2 Mappings of two-systems which are orthogonal to the unit circle but which are not self-polar.

$$q_1 = \alpha + (\alpha^2 - 1)^{\frac{1}{2}} \quad , \quad q_2 = \alpha - (\alpha^2 - 1)^{\frac{1}{2}}. \tag{29}$$

Since the principal screws meet at right angles and

$$q_1' = q_1^{-1} = q_2 \quad , \quad q_2' = q_2^{-1} = q_1 \tag{30}$$

then under the elliptic polarity, the principal screws are interchanged and thus the cylindroid may be transformed by a right angle rotation about its nodal axis. This example demonstrates why it is necessary to adopt the extended viewpoint of interpreting points and circles on the (h, r) plane as pencils of screws and cylindroids when examining the properties of induced transformations.

Finally, it is useful to consider in Euclidean space, the second special two-system formed by screws with a constant pitch h_α that are parallel to the z axis and lie in the x-z plane. This two-system maps onto a straight line $h = h$ in the (h, r) plane (r is measured in the x direction). The polar two-system maps onto a circle, tangent to the origin as shown in Fig. 4.4.3, where θ is determined by (5) and may be used to locate corresponding polar pairs. Under the elliptic polarity, the line interval from $(h_\alpha, 0)$ to $(h_\alpha, +\infty) = (\infty, \infty)$ is continuously transformed onto the lower semi-circle from $(1/h_\alpha, 0)$ to $(0, 0)$. Similarly, the line interval from $(h_\alpha, -\infty) = (\infty, \infty)$ is mapped onto the upper semi-circle from $(0, 0)$ to $(1/h_\alpha, 0)$. (The symbols $\pm\infty$ are used to indicate direction.)

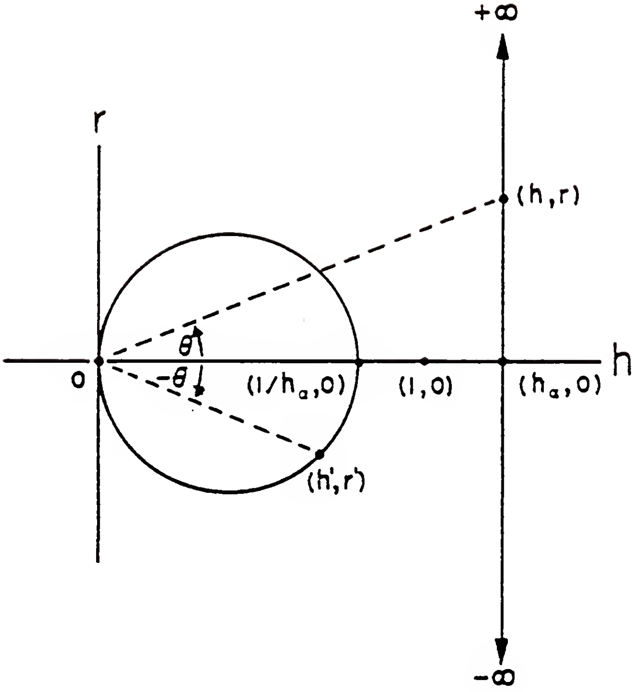

Figure 4.4.3 A mapping of the second special two-system and its elliptic polar.

In considering the elliptic polarity of screws in Euclidean space, it was always necessary to exclude translations from consideration for reasons that are detailed in Section 4.1. However, it is interesting to explain the effect of translation on the elliptic polar of a screw. Rather than considering translations of space as in the previous sections, it is slightly more convenient to consider a translation of the origin which is equivalent. The parallel two-system is now used to describe origin translations.

Figure 4.4.4 illustrates a screw p belonging to the parallel two-system together with a cylindroid centered on the origin 0 which is generated by the elliptic polars of the parallel two-system. The elliptic polar of p is given by p' on this cylindroid. Consider that 0 translates in the x direction without passing through the axis of p (this simplifies the discussion) and the polar of p is now p_1'. The parallel two-system now maps to an identical cylindroid centered on 0_1 which can be obtained by a rigid body translation of the cylindroid from 0. However, the polars p' and p_1' of the screw p, occupy different positions on their respective cylindroids and in general have distinct pitches. Consider now a translation parallel to the axis of p from 0_1 to 0_2 which also translates the polar cylindroid as a rigid body. The elliptic polar screw p_1' is transformed into p_2' which both have identical pitches and positions on their respective cylindroids. Finally, 0_2 is moved to 0_3 by a rotation ϕ about the axis of p which also rigidly carries the

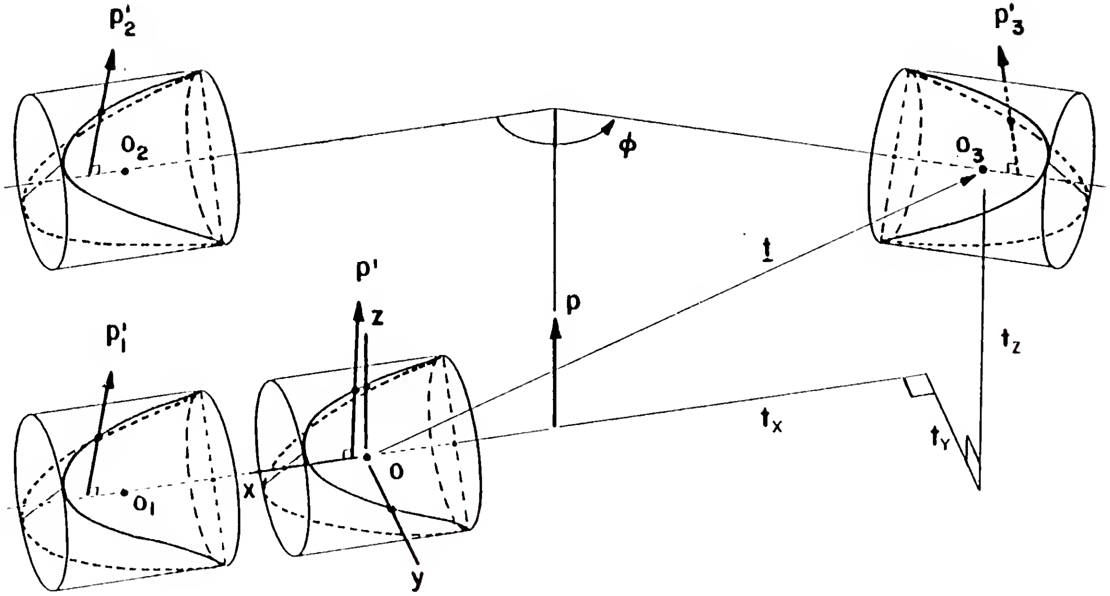

Figure 4.4.4 The effect of an origin translations on the elliptic polar of a screw.

cylindroid to its final position and p_2', p_3' have identical pitches and positions on their respective cylindroids. (It is noted that the cylindroid of 0_3 no longer corresponds to the original two-system but to one that has also been rotated about p by the same angle ϕ.) Thus the change of pitch and location of the polar of a single screw p due to the translation $\underline{t}$ from 0 to 0_3 has been determined by the sequence of a translation perpendicular to the screw axis followed by a translation parallel to the screw axis, and then by a rotation about the screw axis.

Algebraically, the effect of the origin translation in this manner from 0 to 0_3 on the polar of a screw can be formulated by,

$$p_3' = (\hat{\Delta}\ \hat{K}_t^{-1}\ \hat{\Delta})p' . \tag{31}$$

In (31), the first application of $\hat{\Delta}$ transforms p' into p for origin 0. The collineation K_t^{-1} then expresses p with respect to 0_3 and the final application of $\hat{\Delta}$ yields the polar of p with respect to 0_3. Using the tetrahedron relationship

$$K^T\ \tilde{\Delta}\ \hat{K} = \tilde{\Delta} \tag{32}$$

(31) may be expressed as

$$p_3' = \hat{k}_t^{-1}\ p' \tag{33}$$

where $\hat{k} = \hat{K}^{-T}$ is the corresponding transformation for axis coordinates. As discussed in Section 4.1, this is clearly not the same as a direct origin translation from 0 to 0_3 for which p' transforms as does any other screw

$$p_3' = \hat{K}_t^{-1} p' \tag{34}$$

Clearly, the two screws p_3' in (33) and (34) are distinct. This exemplifies the problem of admitting origin translations in connection with the elliptic polarity of screws.

CHAPTER 5

APPLICATIONS

> In general the position as regards all such new calculi is this--That one cannot accomplish by them anything that could not be accomplished without them. However, the advantage is, that, provided such a calculus corresponds to the inmost nature of frequent needs, anyone who masters it thoroughly is able--without the unconscious inspiration of genius which no one can command--to solve the respective problems, yea, to solve them mechanically in complicated cases in which, without such aid, even genius becomes powerless. Such is the case with the invention of general algebra, with the differential calculus, and in a more limited region with Lagrange´s calculus of variations, with my calculus of congruences, and with Mobius´s calculus. Such conceptions unite, as it were, into an organic whole countless problems which otherwise would remain isolated and require for their separate solution more or less application of inventive genius. C.F. Gauss [1843, p. 298]

Screw theory is used in this chapter for the quantification of the freedom and constraint of rigid bodies and is applied to the analysis of serial robotic manipulators. Using screw theory, a current type of hybrid control is shown to be noninvariant and alternative invariant formulations are presented.

In order to refer concisely to a specific area of mechanics, it is useful to introduce the word "kinestatics" which clearly is derived from kinematics and statics. However, a single word to describe these two concepts is of no great advantage here and instead the word is given a much more specific meaning. Kinestatics is defined as the study of the dualistic properties and relations of first order kinematics and statics for massless rigid bodies. To a lesser extent zero and second order kinematics are also included as far as they pertain to the characterization of first order kinematics and statics.

It should be noted that Phillips [1984] has recognized the desirability of having such a term to describe the dualistic relations of kinematics and statics. However, he uses the previously established word "kinetostatics" for this purpose. This term usually refers to a specific area in dynamics, Whittaker [1904], Federhofer [1953], Sandor and Erdman [1984]. Briefly, when the position, velocity and acceleration of a system of rigid bodies is known, it is then possible to introduce D´Alembert´s force and to determine the reaction forces and driving forces using the same methods as for static equilibrium. This is usually referred to as a kinetostatic analysis. Phillips uses this term in connection with massless bodies to apparently describe what is referred to here as kinestatics.

It may also be useful to note that the terms "noninvariant" and "noninvariance" are used here. They may appear to be double negatives when compared to the terms "variant" and "variance" (which is used in statistics). However, the mathematical term which is relevant here is "invariant" and thus the negation of this term has a precise mathematical meaning which is the intention here.

Section 5.1 reviews the concepts involved for the hybrid control of a manipulator where it is necessary to simultaneously control the end-effector twist and reaction wrench. The relevance of a kinestatic filter is discussed in altering specified manipulator commands to insure that they are consistent with the environment. A general scheme for hybrid control is presented in terms of kinestatics. The role of a control strategy is considered in the context of combining commands related directly or indirectly to the end-effector twist and reaction wrench.

A current type of noninvariant hybrid control is analyzed in Section 5.2. Using screw theory and the previous developments of elliptic geometry, it is shown that the theory yields noninvariant results with respect to Euclidean translations and changes in unit length.

The analysis of the noninvariant theory is continued in Section 5.3 where it is considered in terms of minimization principles. A simple numerical example is presented which clearly demonstrates the noninvariance of the solution using a five degree-of-freedom manipulator.

Finally, Section 5.4 presents alternative invariant filters which may be used in the hybrid control of a manipulator and their areas of applicability are detailed. This section also indicates that there are many more possible solutions which may be formulated by optimizing invariant criteria.

Section 5.1 Hybrid Control

> There was once a rigid body which lay peacefully at rest. A committee of natural philosophers was appointed to make an experimental and rational inquiry into the dynamics of that body. The committee received special instructions. They were to find out why the body remained at rest, notwithstanding that certain forces were in action. . . .
>
> The committee was judiciously chosen. Mr. Anharmonic undertook the geometry. He was found to be of the utmost value in the more delicate parts of the work, though his colleagues thought him rather prosy at times. He was much aided by his two friends, Mr. One-to-One, who had charge of the homographic department, and Mr. Helix, whose labours will be seen to be of much importance. As a most respectable, if rather old-fashioned member, Mr. Cartesian was added to the committee, but his antiquated tactics were quite outmanoeuvred by those of Helix and One-to-One. I need only mention two more names. Mr. Commonsense was, of course, present as an ex-officio member, and valuable service was rendered even by Mr. Querulous, who objected at first to serve on the committee at all. He said that the inquiry was all nonsense, because everybody knew as much as they wished to know about the dynamics of a rigid body. The subject was as old as the hills, and had all been settled long ago. He was persuaded, however, to look in occasionally. It will appear that a remarkable result of the labours of the committee was the conversion of Mr. Querulous himself.
>
> The committee assembled in the presence of the rigid body to commence their memorable labours. There was the body at rest, a huge amorphous mass, with no regularity in its shape--no uniformity in its texture. But what chiefly alarmed the committee was the bewildering nature of the constraints by which the movements of the body were hampered. They had been accustomed to nice mechanical problems, in which a smooth body lay on a smooth table, or a wheel rotated on an axle, or a body rotated around a point. In all these cases the constraints were of a simple character, and the possible movements of the body were obvious. But the constraints in the present case were of puzzling complexity. There were cords and links, moving axes, surfaces with which the body lay in contact, and many other

> geometrical constraints. Experience of ordinary problems in mechanics would be of little avail. In fact, the chairman truly appreciated the situation when he said, that the constraints were of a perfectly general type. R.S. Ball [1887, pp. 496-497]

The hybrid control of a robot manipulator enables simultaneous control of the manipulator motion and its interactions with external constraints. Historically, the hybrid control of manipulators has its roots in position and velocity control as well as in force and torque control, Whitney [1985]. In circumstances where a manipulator must execute kinematically constrained motion, some form of hybrid control may be essential. Important applications include insertion tasks, contour tracking and coordinating multiple robots grasping a single workpiece.

For simplicity, the discussion here considers only the kinestatic aspects of hybrid control in order to exploit the elegant duality afforded by screw theory. Thus in this sense, the hybrid control of a manipulator involves controlling the instantaneous twist of the end-effector as well as controlling the interaction wrenches resulting from applied loads and constraints of the environment acting on the end-effector and its workpiece. All physical elements are assumed to be massless for this presentation and thus dynamic relations are precluded.

It is assumed that the manipulator is firmly grasping a workpiece which is modeled as a rigid body and that all interactions with the environment take place through the workpiece. As suggested by Mason [1978], it is useful to initially consider hybrid control in two extreme circumstances. In the first case, the workpiece is in free space and is thus unconstrained since it lacks any contact with the environment. Generally, it is then possible to control the six independent velocity degrees-of-freedom for the body. That is, it is possible for the body to twist about any of the ∞^5 screws in space with any of ∞^1 possible twist magnitudes. The freedom of the body may also be described by six linearly independent screws which form a basis for a 6-system of twists. Every possible twist may be formed as a unique linear combination of the basis. On the other hand, however, it is impossible for the manipulator to apply any wrench to the body since there is no constraining environment which can produce the reaction wrench necessary for static equilibrium. Thus the totality of possible applied or reaction wrenches is empty and may be referred to as a 0-system of wrenches. Clearly, any command given to the manipulator to apply a nonzero wrench is erroneous since its specification is inconsistent with the environment. For this case pure twist control is appropriate.

The second case is at the opposite extreme where it is assumed that the workpiece is rigidly attached to the fixed environment such that it is completely constrained. Generally, it is then possible for the manipulator to apply any wrench to the body since the environment is able to provide the necessary reaction wrench for static equilibrium. The totality of such applied or reaction wrenches forms a 6-system of wrenches. Every possible wrench may be described as a unique linear combination of six linearly independent screws which form a basis for the 6-system. Since the workpiece is fully constrained, it is unable to twist about any screw and thus its freedom is described by a 0-system. Clearly, any command given to the manipulator to execute a nonzero twist must be erroneous since its specification is inconsistent with the environment. In this case pure wrench control is appropriate for the manipulator.

Intermediate between the two extremes of total freedom and total constraint is when the environment partially constrains the workpiece and thus only constrained motion is possible. It is assumed that due to the contraint of the environment that the body has n degrees-of-freedom. For the most general case $0 \leq n \leq 6$ but for constrained motion $1 \leq n \leq 5$. Using axis coordinates, let T be any twist of the n-system and let $B_1 \ldots B_n$ be n linearly independent screws which span the system then

$$T = B \lambda \tag{1}$$

where B is the $6 \times n$ matrix

$$B = [B_1 \ . \ . \ . \ B_n] \tag{2}$$

and where λ is an $n \times 1$ column array of scalar multipliers. Equation (1) is referred to as the freedom equation of the n-system of twists.

Suppose a wrench is applied to a rigid body whose freedom is described by an n-system. If for all possible twists of the body the virtual power (instantaneous virtual work) due to the applied wrench vanishes, then the wrench is said to be a constaint wrench. Expressing the wrench w in ray coordinates, then as shown in Section 3.4 the virtual work is given by

$$T^T w = 0 \tag{3}$$

and the twist T and wrench w are said to be reciprocal. Essentially, this is a projective relation between their screws. For w to be reciprocal to every T in the n-system then it is necessary and sufficient that

$$B^T w = 0. \quad (4$$

The six components of the constraint wrench w are related by n independent equations in (4) and thus the totality of such wrenches require

$$m = 6 - n \tag{5}$$

independent quantities for their specification. Let $a_1 \ldots a_m$ be m linearly independent screws which are each reciprocal to the n-system then

$$B^T a = 0 \tag{6}$$

where a is the $6 \times m$ matrix

$$a = [a_1 \; . \; . \; . \; a_m]. \tag{7}$$

The m-system of constaint wrenches may be expressed in the form

$$w = a \; \varepsilon \tag{8}$$

where ε is a $m \times 1$ array of scalar multipliers. Equation (8) is referred to as the freedom equation of the m-system of wrenches. It should be noted that (4) also expresses the same system and is referred to as the constraint equation for the m-system of wrenches. Further, since every twist T is reciprocal to the m-system of wrenches then

$$a^T \; T = 0 \tag{9}$$

represents the constraint equation of the n-system of twists.

In both freedom equations (1) and (8) the scalar multipliers λ and ε may take on any sets of values. This preclues any twist or wrench acting unilaterally, i.e. in only one sense. For example, a sphere tangent to a plane may not twist (translate) towards the plane due to a constraint wrench (force) at the point of contact. However, a motion in the opposite direction is unconstrained. The twist and wrench are both said to act unilaterally. For simplicity, unilateral action is excluded here. An interesting theory of such twists and wrenches has been presented by Ohwovoriole [1980], Ohwovoriole and Roth [1981].

Consider now that the manipulator workpiece is able to execute constrained motion. Two general cases may be described. In the first case, the specified twist T of the body is in the n-system of B which represents the freedom of the workpiece. Reciprocally, the specified applied wrench w (or reaction wrench -w) of the body is in the m-system of a which represents the constraint of the workpiece. Thus the specified twist and wrench of the workpiece are both consistent with the environment which is modeled kinestatically by matrices B and a. Under these conditions it is theoretically feasible to control the workpiece with twist T and wrench w and hybrid control of the manipulator is applicable. The general methodology of enforcing such a simultaneous control is embodied in the control strategy for which various types are described subsequently.

The problem which is primarily addressed in this chapter is the second case where the specified twist T and (and/or) wrench w are not consistent with the environment. It may then be desirable to alter or filter T and w in order to make them consistent with the environment and thus reduce the second case to the first which is amenable to a form of hybrid control. Specifically, T is filtered such that it is in the n-system of B and w is filtered such that it is in the m-system of a. If an erroneous specification of T and w is not filtered, then the control strategy will attempt to effect a control that is physically impossible. The outcome of such an action is dependent upon the details of the control strategy, the automatic control compensation and physical considerations such as saturation of system components and friction. It may result in a degradation of accuracy and performance for small errors in T and w to possible instability.

If a model of the environment is available in terms of B and a, then an erroneous specification of T and w is easily recognized since generally

$$B^T w \neq 0, \quad a^T T \neq 0. \tag{10}$$

However, once such an error is detected what is a logical course of action? One solution is to simply halt operation of the system and communicate the problem to a higher device. As an alternative in some applications, the methodology considered here is to create a kinestatic filter based upon invariant criteria which is in general nonunique. The filter merely alters the specified twist and wrench such that they are consistent with the kinestatic model of the environment and is detailed in Section 5.4. Whether it is logical to employ such a filter at all is dependent upon the particular application and the range of errors involved.

Figure 5.1.1 illustrates a general scheme for a manipulator controlling a workpiece using a possible hybrid control system. The desired twist T and wrench w of the workpiece are specified by a kinestatic planner which may be a data

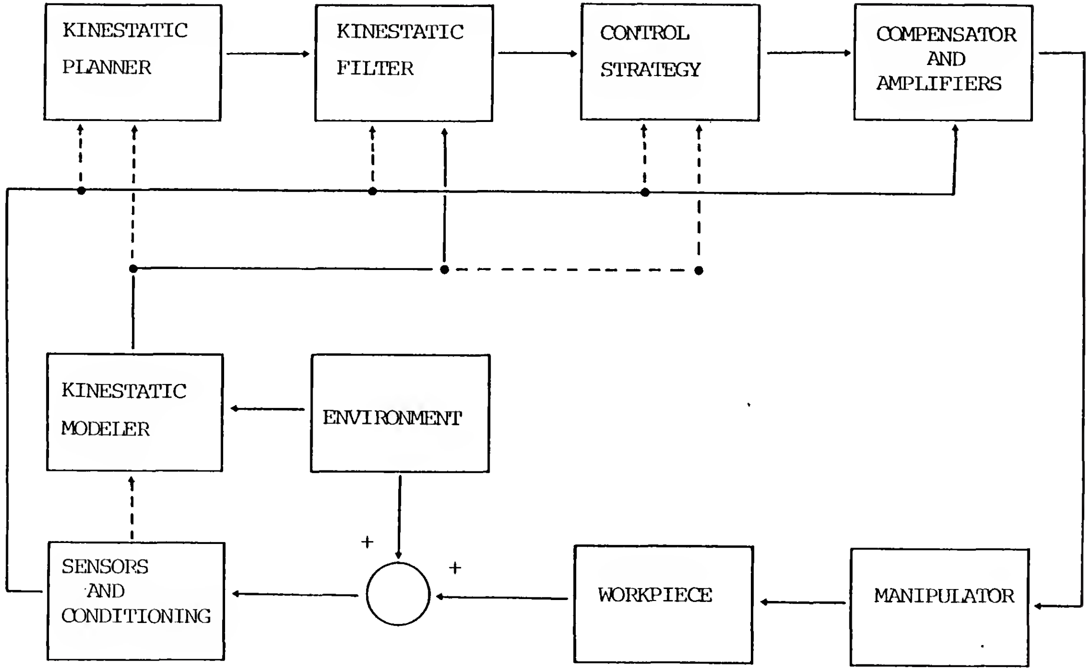

Figure 5.1.1 A general scheme for the hybrid control of a manipulator.

file programmed off-line, direct human input using a joystick or keyboard, an on-line artificial intelligence program based on sensor feedback, etc. The kinestatic filter uses a model of the environment to alter the specified twist and wrench if necessary. The kinestatic model may be either determined a priori or determined on-line using sensor data. Based on the filtered commands and possibly on sensory feedback, the control strategy enforces an algorithm for simultaneous twist and wrench control. The resulting signal is then input to the automatic control system which includes dynamic considerations. The action of the manipulator and workpiece in the environment is determined by high and low level sensing. Although not shown in the figure, it may also be desirable to filter some or all of the sensor data. If the kinestatic model is formed from redundant sensory input or from accurate a priori data then kinestatic filtering of the sensor data becomes feasible.

In order to illustrate the role of the control strategy two representative cases are examined, one which is designed for joint space control and one which is designed for end-effector or operational space control. For simplicity, a six degree-of-freedom manipulator is employed throughout which is assumed to be in a nonsingular configuration.

In order to describe the joint space control strategy, it is first necessary to determine the relationships between the desired end-effector twist and wrench and the angular speeds and torques of the manipulator joints. The six joints of the manipulator span a 6-system of screws. The instantaneous twist T of the end-effector is given by the freedom equation,

$$T = J\,\omega \tag{11}$$

where ω is a 6×1 array of joint speeds and J represents the axis coordinates of the unit screw vectors for the joints,

$$J = [J_1 \; . \; . \; . \; J_6]. \tag{12}$$

In most situations the joints are either revolute for which J_i is a unit line vector and ω_i is the angular speed, or prismatic for which J_i is a unit free line vector and ω_i is the rectilinear speed. For brevity, the manipulator is usually referred to as if it is composed exclusively of revolute joints.

The joint torques may be determined by considering the manipulator to be in static equilibrium under the impressed wrench w while twisting about T. Assume initially that all but the ith joint is fixed, then equating the virtual power at the end-effector with that at the joint yields,

$$(J_i\,\omega_i)^T\,w = \tau_i\,\omega_i \tag{13}$$

where τ_i is the joint torque or joint force if it is prismatic. Cancelling the common factor ω_i and repeating for $i = 1 \; . \; . \; . \; 6$ gives

$$\tau = J^T w. \tag{14}$$

Therefore, in order for the manipulator end-effector to twist about T while being in static equilibrium under the applied wrench w, it is necessary for the ith joint actuator to have the angular speed ω_i while exerting torque τ_i. However, joint actuators are single degree-of-freedom devices and thus cannot follow two independent commands. Although this is an obvious conclusion, a simple illustration is useful in making this point explicit.

For example, suppose the actuator is a permanent magnetic DC motor. Using the model in Electro-Craft [1980, p. 4.15] then the various transfer functions expressed in terms of the Laplace transform variable s are:

$$\tau(s) = K_T \, I(s) \tag{15}$$

$$\omega(s) = \frac{K_T \, V_{(s)}}{(sJ + D)(sL + R) + K_E \, K_T} \tag{16}$$

$$I(s) = \frac{(sJ + D) \, V_{(s)}}{(sJ + D)(sL + R) + K_E \, K_T} \tag{17}$$

$$\omega(s) = \frac{\tau(s)}{(sJ + D)} \tag{18}$$

where the variables are:

$\tau(s)$ - torque

$\omega(s)$ - speed

V(s) - applied voltage

I(s) - applied or armature current

and the constants are:

K_T - torque constant

K_E - voltage constant

J - armature inertia

D - armature damping

L - armature inductance

R - motor resistance.

As indicated by (15) and (16), torque is often controlled by a current amplifier whereas speed is usually controlled by a voltage amplifier. Clearly, current and voltage cannot be controlled independently since they are coupled by the motor dynamics (17) as are the speed and torque (18).

For control in the joint space the control strategy involves the imposition of a relation between speed and torque. For example, linear schemes based on damping or inverse damping may be developed for each individual joint using a diagonal matrix,

$$\tau_g = \tilde{g}_1 \, \omega \tag{19}$$

or

$$\omega_G = \tilde{G}_2 \, \tau \tag{20}$$

where $\tilde{g}_1$ and $\tilde{G}_2$ are not necessarily nonsingular. The tilda is employed to denote that the transformation is between dual variables which is analogous to that for twists and wrenches. The damping scheme (19) indicates that speed is transformed into torque and that torque control is then used to derive the actuators. The command torque is often a linear combination of τ_g and the static torque τ. Analogously, the inverse damping scheme transforms torque into speed and speed control is used to drive the actuator. The commanded speed is usually a linear combination of ω_G and ω.

Other joint based control strategies are possible. For example, early force reflecting teleoperator systems were based on joint compliance and stiffness, Corliss and Johnson [1967]. A recent review that includes joint control strategies is given by Whitney [1985].

It should be noted that there is one case of exception where the end-effector may be driven with twist T while simultaneously equilibrating the applied wrench w without enforcing a relation between the joint speeds ω and the joint torques τ. Assume that the Jacobian may be partitioned into two reciprocal subspaces and

$$J = J_T \; J_W \;, \quad \omega = \begin{bmatrix} \omega_T \\ \omega_W \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_T \\ \tau_W \end{bmatrix} \tag{21}$$

where

$$J_T^T \, J_W = 0. \tag{22}$$

If T is in the subspace of J_T then,

$$T = J_T \, \omega_T \text{ and } \omega_w = 0. \tag{23}$$

Since w is reciprocal to T then,

$$\tau_w = J_w^T \, w \text{ and } \tau_T = 0. \tag{24}$$

In (23) only the joints corresponding to ω_T may have nonzero speeds while those of ω_w are stationary. From (24) these stationary joints must maintain the torques τ_w while the remaining joints corresponding to τ_T require no actuator torques since the actuator bearings are providing the proper reactions for equilibrium. Thus each manipulator joint is either speed controlled for ω_T or torque controlled for τ_w. An elementary example is given by a Cartesian manipulator whose first three joints are prismatic at right angles J_w and whose last three joints are revolutes J_T which cointersect at a wrist point. The manipulator is grasping a sphere whose center is coincident with the wrist point. The specified twist T is a rotation through the sphere center in which it is necessary to drive the last three joints with speeds ω_T while the first three joints are stationary, $\omega_w = 0$. The applied wrench w is a force on the sphere through the wrist point and thus produces no moment about the last three joints such that equilibrium is maintained by the bearings and $\tau_T = 0$. At the first three joints the force is equilibrated by the actuator force levels τ_w.

The second manner in which a control strategy may be generally implemented is in Euclidean, end-effector or operational space. That is, the control strategy imposes a relation between the end-effector twist T and the applied or reaction wrench w. For example, linear schemes based on damping or inverse damping may be formulated by

$$w_d = \tilde{d}_1 \, T \tag{25}$$

$$T_D = \tilde{D}_2 \, w \tag{26}$$

where $\tilde{d}_1$, $\tilde{D}_2$ are not necessarily nonsingular and are often diagonal. It should be noted that the matrices $\tilde{d}_1$ and $\tilde{D}_2$ introduce a change in space element. However, they are in general not projective correlations of three-dimensional space but are homographic correlations as defined in Section 2.3. Assuming in (25), (26) that the matrices $\tilde{d}_1$, $\tilde{D}_2$ are diagonal, then these relations effect a damping action along and about the three coordinate axes. This is equivalent to three linear and three rotary dampers through a center of damping that is coincident with the origin. The position and orientation of the damping system may be easily altered by introducing a Euclidean collineation of space $\hat{K} = \hat{k}^{-T}$, Section 3.3. Substituting $\hat{K}w$, $\hat{k}T$ for w, T in (25) and (26) yields,

$$w_f = \tilde{f}_1 \, T = (\hat{k}^T \, \tilde{d}_1 \, \hat{k}) \, T \tag{27}$$

$$T_F = \tilde{F}_2 \, w = (\hat{K}^T \, \tilde{D}_2 \, \hat{K}) \, w. \tag{28}$$

Whitney [1977] has employed a damping system control strategy in conjunction with resolved motion rate control which together he called accomodation. The twist T and wrench w are expressed in terms of a coordinate frame attached to the end-effector and the relation (26) is used in the feedback loop of the control system. (In terms of a fixed coordinate (28) is also appropriate with a variable $\hat{K}$.) Loosely speaking, the effect of the control strategy is analogous to moving a specially shaped object through a homogeneous viscous fluid.

More general control strategies in operational space are possible. For example, Salisbury [1980] employs an active stiffness control strategy for control of a mechanical hand. Dynamic control strategies include Khatib [1983] and Hogan [1985]. Further techniques are also reviewed by Whitney [1985].

Finally, it is also useful to demonstrate that (25), (26) may be expressed in terms of the joint space. Multiplying them resepctively by J^T and J^{-1} and using (11), (14) yields,

$$\tau_d = (J^T \tilde{d}_1 J)\ \omega \tag{29}$$

$$\omega_D = (J^{-1} \tilde{D}_2 J^{-1})\ \tau . \tag{30}$$

Analogously, the relations in joint space (19), (20) may also be expressed in Euclidean space. Multiplying the equations respectively by J^{-T} and J yields,

$$w_g = (J^{-T} \tilde{g}_1 J^{-1}) T \qquad (31)$$

$$T_G = (J \tilde{G}_2 J^T) w. \qquad (32)$$

When J is noninvertible, it is noted that (29) and (32) remain useful.

Section 5.2 Noninvariant Methods

> It was therefore resolved to make a series of experiments with the simple object of seeing how the body could be moved. Mr. Cartesian, having a reputation for such work, was requested to undertake the inquiry and to report to the committee. Cartesian commenced operations in accordance with the well-known traditions of his craft. He erected a cumbrous apparatus which he called his three rectangular axes.
>
> . . . Cartesian was a very skillful calculator, and by a series of experiments even with his unsympathetic apparatus he obtained some knowledge of the subject, sufficient for purposes in which a vivid comprehension of the whole was not required. The inadequacy of Cartesian's geometry was painfully evident when he reported to the committee on the mobility of the rigid body. 'I find,' he said, 'that the body is unable to move parallel to x, or to y, or to z; neither can I make it rotate around x, or y, or z; but I could push it an inch parallel to x, provided that at the same time I pushed it a foot parallel to y and a yard backwards parallel to z, and that it was also turned a degree around x, half a degree the other way around y, and twenty-three minutes and ninteeen seconds around z.'
>
> 'Is that all?' asks the chairman. 'Oh, no,' replied Mr. Cartesian, 'there are other proportions in which the ingredients may be combined so as to produce a possible movement,' and he was proceeding to state them when Mr. Commonsense interposed. 'Stop! stop!' said he, 'I can make nothing of all these figures. This jargon about x, y, and z may suffice for your calculations, but it fails to convey to my mind any clear or concise notion of the movements which the body is free to make,'

> Many of the committee sympathised with this view of Commonsense, and they came to the conclusion that there was nothing to be extracted from poor old Cartesian and his axes. They felt that there must be some better method, and their hopes of discovering it were raised when they saw Mr. Helix volunteer his services and advance to the rigid body. . . .
>
> Let the constraints be what they may--let the position B lie anywhere in the close neighborhood of A--Helix found that he could move the body from A to B by an extremely simple operation. With the aid of a skilful mechanic he prepared a screw with a suitable pitch, and adjusted this screw in a definite position. The rigid body was then attached by rigid bonds to a nut on this screw, and it was found that the movement of the body from A to B could be effected by simply turning the nut on the screw. A perfectly definite fact about the mobility of the body had thus been ascertained. It was able to twist to and fro on a certain screw.
>
> Mr. Querulous could not see that there was any simplicity or geometrical clearness in the notion of a screwing movement; in fact he thought it was the reverse of simple. Did not the screwing movement mean a translation parallel to an axis and a rotation around that axis? Was it not better to think of the rotation and the translation separately than to jumble together two things so totally distinct into a composite notion?
>
> But Querulous was instantly answered by One-to-One. 'Lamentable, indeed,' said he, 'would be a divorce between the rotation and the translation. Together they form the unit of rigid movement. Nature herself has wedded them, and the fruits of their union are both abundant and beautiful.' R.S. Ball [1897, pp. 497-499]

In this section a type of hybrid control is examined which has recently become the subject of growing research. It was apparently first formulated by Mason [1978, 1981, 1982] with early experimental work by Craig and Raibert [1979] and Raibert and Craig [1981]. More recent work in the area includes West and Asada [1985], Zhang and

Paul [1985] as well as a related paper by Kobayashi [1985]. The theory presented in these works is referred to here as noninvariant hybrid control theory or more briefly as the noninvariant theory for reasons which are presented subsequently.

The noninvariant theory deals with the development of a control strategy for constrained motion which is referred to as "compliant" motion since the manipulator must "comply" with the constraints of the environment. From a non-engineering standpoint "compliance" is a descriptive term for this type of action although from an engineering viewpoint, it is somewhat of a misnomer. As used here, compliance has a precise meaning indicating the inverse of stiffness. Compliant control is used to indicate a control strategy which involves the transformation of force information into positional information and may be done either actively or passively. For example, passive compliance is used in the remote center of compliance device, see Whitney [1982] for a recent account.

In general, constrained motion of a rigid body refers to the freedom of the body subject to complex constraints which may be rigid or nonrigid, active or passive, working or nonworking, bilateral or unilateral, holonomic or nonholonomic, etc. However, the interest here is in kinestatics which deals with the dualistic relations of first order kinematics and statics for massless rigid bodies. The constraints considered here are limited to rigid, workless, bilateral constraints that are generally expressed as nonholonomic equations since they involve spatial angular velocities.

The kinestatic aspects of noninvariant hybrid control theory are developed here using the screw theory developed in the previous chapter. No attempt is made to present an explanation that corresponds to the original development in every detail. This is because the original development of the noninvariant theory is a highly abstract yet imprecisely defined formalism which is based primarily on heuristic notions. What is missing is a general systematic development of the equations necessary to implement the control which is analytically and geometrically rigorous. In place of such a methodology simple examples are presented which appeal to the reader's intuition and thus appear to support the advocated theory. This is not meant to be disparaging, but is rather to emphasize that more satisfactory results may be obtained by an alternative approach. The viewpoint here is to consider the noninvariant theory in terms of kinestatic filtering. This viewpoint is apparently compatible with the one by West and Asada [1985] who give an elegant, systematic and general development employing generalized inverses and projector matrices. However, since this work is a generalization of the original theory it retains the same problem of noninvariance along with its propagation in respect to the (1, 2) generalized inverse.

Following the development in the previous section, it is assumed that the manipulator is grasping a workpiece which is modeled as a rigid body and that all interactions with the environment take place through the workpiece. The analysis is initially limited to considering the constrained motion of the workpiece and is then extended to include the control of the manipulator in the following section.

Due to the constraint of the environment the workpiece is assumed to have n degrees-of-freedom. If the twist T represents any possible motion of the body and w represents any possible reaction wrench, then from Section 5.1 the freedom equations of w and T are

$$w = a\ \varepsilon \tag{1}$$

$$T = B\ \lambda \tag{2}$$

and the constraint equations of w and T are

$$B^T\ w = 0 \tag{3}$$

$$a^T\ T = 0 \tag{4}$$

where

$$a = [a_1\ .\ .\ .\ a_m] \tag{5}$$

$$B = [B_1\ .\ .\ .\ B_n] \tag{6}$$

$$m = 6 - n. \tag{7}$$

The screws of a and B respectively are said to form an m-system and n-system. Since w and T must be reciprocal,

$$T^T w = 0 \tag{8}$$

and it follows that the screw systems a and B are reciprocal,

$$a^T B = 0. \tag{9}$$

If a planner specifies a wrench w that is in the m-system a, and a twist T which is in the n-system B, then the desired motion and constraint of the workpiece is consistent with the environment as kinestatically modeled by a and B. Theoretically, the constrained motion may be executed by a suitable control strategy. This problem is not considered here further. Instead, the problem developed here is when the specified wrench w and (and/or) twist T are not consistent with the environment. This is expressed in terms of the freedom equations by

$$w \neq a\,\varepsilon\ ,\ T \neq B\,\lambda \tag{10}$$

and the constraint equations by

$$B^T w \neq 0\ ,\ a^T T \neq 0. \tag{11}$$

It may then be desirable to alter or filter w and T such that they are consistent with the environment. This transforms the problem into the previous one which is amenable to implementation by an appropriate control strategy. The noninvariant filtering which forms the basis of noninvariant hybrid control theory is now developed using kinestatic screw theory.

As detailed in Section 4.1, the noninvariance problem of hybrid control is based upon a misinterpretation of the equation for the vanishing of virtual work in (8), $T^T w = 0$. The correct interpretation of (8) is that the twist T in axis coordinates is reciprocal to the wrench w in ray coordinates which is essentially a projective relationship between their screws. The incorrect interpretation of (8) is that T and w are "orthogonal" since they satisfy a relation that appears similar to a "standard" inner product. This is the fundamental error in noninvariant hybrid control theory. Because of this misinterpretation, the noninvariant theory may be explained using the elliptic geometry of screws presented in Section 3.2.

The elliptic polarity of screws is represented in correlation form by the identity matrix for both ray coordinates and axis coordinates,

$$\tilde{\Gamma}_1 = \tilde{I}_6 \quad \text{and} \quad \tilde{\gamma}_1 = \tilde{I}_6. \tag{12}$$

The elliptic polars of the m-system a and the n-system B are given by

$$A' = \tilde{\Gamma}_1 \, a = \tilde{I}_6 \, a \tag{13}$$

$$b' = \tilde{\gamma}_1 \, B = \tilde{I}_6 \, B \tag{14}$$

where A' is an m-system in axis coordinates and b' is an n-system in ray coordinates. The inverse relations are simply

$$a = \tilde{\gamma}_1 A' = \tilde{I}_6 A' \tag{15}$$

$$B = \tilde{\Gamma}_1 b' = \tilde{I}_6 b' \tag{16}$$

Substituting in turn (15) and (16) in the reciprocal relation (9) yields

$$A'^T \tilde{\gamma}_1 B = A'^T \tilde{I}_6 B = 0 \tag{17}$$

$$a^T \tilde{\Gamma}_1 b' = a^T \tilde{I}_6 b' = 0 \tag{18}$$

and according to Section 3.2, the screws of A' and B are said to be elliptic conjugates or elliptic-orthogonal as are the screws of a and b'. For (17), (18) to be logically consistent it is necessary to include the elliptic polarity since they are bilinear forms of matrices. However, in terms of the noninvariant theory the identity matrices are suppressed and they take the forms,

$$A'^T B = 0 \tag{19}$$

$$a^T b' = 0. \tag{20}$$

The systems A' and B are termed "orthogonal" as are a and b'.

Figure 5.2.1 symmetrically illustrates the four screw systems as vertices of a tetrahedron with similar interrelationahips expressed on opposite edges. (Actually it is even more elegant to define A, B', a', b using $\tilde{\Delta}$ and employ ray coordinates for the vertices and axis coordinates for the

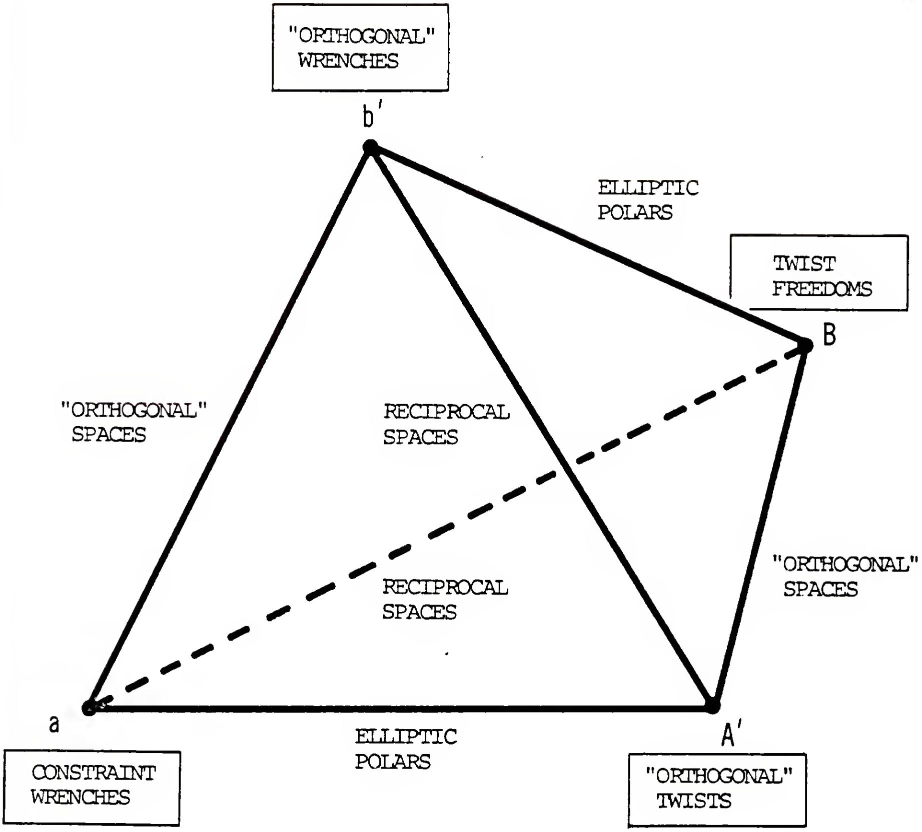

Figure 5.2.1 Reciprocal and "orthogonal" screws systems.

faces.) For brevity, the systems A' and b' are referred to as the "orthogonal" twists and "orthogonal" wrenches. Using (13), (14) in (9) it is readily shown that they are also reciprocal, $A'^T b' = 0$. It is possible to form two distinct bases for the 6-system of screws, one in ray coordinates and the other in axis coordinates,

$$u = [a \; b'] = v' \qquad (21)$$

$$V = [A' \; B] = U' \qquad (22)$$

where the corresponding subspaces are elliptic polars.

It is now supposed that the planner has specified a wrench w and a twist T which are not in the respective subspaces of a and B. Clearly these commands are erroneous since they are not consistent with the kinestatic model of the environment. According to noninvariant hybrid control theory, the specified wrench w and twist T may be altered to be consistent with the environment by taking their "orthogonal" projections on a and B respectively, that is by filtering out the projections on the "orthogonal" spaces b' and A'.

The wrench w and twist T may be expressed in terms of "orthogonal" decompositions using (21), (22) and

$$w = u \, \varepsilon = [a \; b'] \begin{bmatrix} \varepsilon^* \\ \varepsilon^\# \end{bmatrix} \qquad (23)$$

$$T = V \lambda = [A' \quad B] \begin{bmatrix} \lambda^{\#} \\ \lambda^{*} \end{bmatrix} \tag{24}$$

where ε, λ are 6×1, ε^{*}, $\lambda^{\#}$ are $m \times 1$ and $\varepsilon^{\#}$, λ^{*} are $n \times 1$. The values of ε, λ may be determined by forming the elliptic inner products of (23), (24) with (21), (22)

$$u^T \tilde{\Gamma}_1 w = u^T \tilde{\Gamma}_1 u \varepsilon \tag{25}$$

$$V^T \tilde{\gamma}_1 T = V^T \tilde{\gamma}_1 V \lambda \tag{26}$$

or equivalently

$$\begin{bmatrix} a^T \tilde{\Gamma}_1 w \\ b'^T \tilde{\Gamma}_1 w \end{bmatrix} = \begin{bmatrix} a^T \tilde{\Gamma}_1 a & . \\ . & b'^T \tilde{\Gamma}_1 b' \end{bmatrix} \begin{bmatrix} \varepsilon^{*} \\ \varepsilon^{\#} \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} A'^T \tilde{\gamma}_1 T \\ B^T \tilde{\gamma}_1 T \end{bmatrix} = \begin{bmatrix} A'^T \tilde{\gamma}_1 A' & . \\ . & B^T \tilde{\gamma}_1 B \end{bmatrix} \begin{bmatrix} \lambda^{\#} \\ \lambda^{*} \end{bmatrix} \tag{28}$$

which decouples each equation into two. Since the subspaces are all of full column rank and the polarity is positive definite then (27), (28) are solved as

$$\varepsilon^{*} = a^{+} w, \quad \varepsilon^{\#} = b'^{+} w \tag{29}$$

$$\lambda^{\#} = A'^{+} T, \quad \lambda^{*} = B^{+} T \tag{30}$$

where the superscript + denotes the left pseudoinverse. In general, for a matrix α of full column rank and a symmetrical matrix $\tilde{\Psi}$, the left pseudoinverse is defined here by

$$\alpha^{+} = (\alpha^{T} \tilde{\Psi} \alpha)^{-1} \alpha^{T} \tilde{\Psi} \qquad (31)$$

which has the property

$$\alpha^{+} \alpha = I_{6}. \qquad (32)$$

Substituting (29), (30) in (23), (24) yields

$$w^{*} = P_{a} w , w^{\#} = P_{b'} w \qquad (33)$$

$$T^{\#} = p_{A'} T, T^{*} = p_{B} T \qquad (34)$$

where the singular matrices P and p are projector matrices. They are defined in a general form using (31),

$$\rho_{\alpha} = \alpha \alpha^{+} \qquad (35)$$

which has the idempotent property

$$\rho_{\alpha} \rho_{\alpha} = \rho_{\alpha} \qquad (36)$$

and has a rank equivalent to that of α. The projector is also left Ψ-symmetrical since

$$\tilde{\Psi} \rho_{\alpha} = (\tilde{\Psi} \rho_{\alpha})^{T} \qquad (37)$$

The 6×6 projector matrices in (33), (34) are symmetrical and have the properties

$$P_{a} + P_{b'} = I_{6} , P_{a} P_{b'} = 0 \qquad (38)$$

$$p_{A'} + p_B = I_6 \;,\; p_{A'}\, p_B = 0. \tag{39}$$

where $\hat{I}_6$ is the identity collineation. Equations (33) and (34) represent the "orthogonal" decomposition of w onto a and b', and T onto A' and B which are respectively denoted by

$$w = w^* + w^\# \tag{40}$$

$$T = T^\# + T^*. \tag{41}$$

Further developments of pseudoinverses and projectors may be found in Pease [1965], Rao and Mitra [1971], Eisemann [1973], Ben-Israel [1974].

To summarize thus far, the planner specifies a twist T and wrench w that are inconsistent with the environment which is modeled kinestatically by a and B. According to the noninvariant theory T and w are filtered to remove the projections on the "orthogonal" spaces A' and b' to yield,

$$w^* = P_a\, w \tag{42}$$

$$T^* = p_B\, T. \tag{43}$$

The filtered wrench w^* and twist T^* are then passed on to the control strategy for execution. Clearly, w^* and T^* are consistent with the environment as are all screws in the respective m-system a and n-system B. "Orthogonal" projection may be shown to be noninvariant in two distinct manners, in respect to Euclidean translations and in respect to changes in unit length.

To demonstrate that "orthogonal" projection leads to noninvariant results, consider an undetermined nonsingular collineation $\hat{K}$, $\hat{k}$ and

$$w_1 = \hat{K}\, w \quad , \quad u_1 = \hat{K}\, u \tag{44}$$

$$T_1 = \hat{k}\, T \quad , \quad V_1 = \hat{k}\, V \tag{45}$$

where in (44), (45) the subscript 1 denotes a result after the collineation. The decomposition equations (23), (24) become

$$w_1 = u_1\, \varepsilon \tag{46}$$

$$T_1 = V_1\, \lambda . \tag{47}$$

Forming the elliptic inner products analogous to (25), (26) yields

$$u_1^T\, \tilde{\Gamma}_1\, w_1 = u_1^T\, \tilde{\Gamma}_1\, u_1\, \varepsilon \tag{48}$$

$$V_1^T\, \tilde{\gamma}_1\, T_1 = V_1^T\, \tilde{\gamma}_1\, V_1\, \lambda \tag{49}$$

or equivalently

$$\begin{bmatrix} a_1^T\, \tilde{\Gamma}_1\, w_1 \\ b_1'^T\, \tilde{\Gamma}_1\, w_1 \end{bmatrix} = \begin{bmatrix} a_1^T\, \tilde{\Gamma}_1\, a_1 & a_1^T\, \tilde{\Gamma}_1\, b_1' \\ b_1'^T\, \tilde{\Gamma}_1\, a_1 & b_1'^T\, \tilde{\Gamma}_1\, b_1' \end{bmatrix} \begin{bmatrix} \varepsilon^* \\ \varepsilon^\# \end{bmatrix} \tag{50}$$

$$\begin{bmatrix} A'^T\, \tilde{\gamma}_1\, T_1 \\ B_1^T\, \tilde{\gamma}_1\, T_1 \end{bmatrix} = \begin{bmatrix} A_1'^T\, \tilde{\gamma}_1\, A_1' & A_1'^T\, \tilde{\gamma}_1\, B_1 \\ B_1^T\, \tilde{\gamma}_1\, A_1' & B_1^T\, \tilde{\gamma}\, B_1 \end{bmatrix} \begin{bmatrix} \lambda^\# \\ \lambda^* \end{bmatrix} . \tag{51}$$

Unlike (27), (28) these equations no longer necessarily decouple. Nevertheless, solving the top equation in (50) for ε^* and the lower one for $\varepsilon^\#$ and analogously in (51) for $\lambda^\#$ and λ^* yields

$$\varepsilon^* = a_1^+ w_1 - a_1^+ b_1' \varepsilon^\# \tag{52}$$

$$\varepsilon^\# = b_1'^+ w_1 - b_1'^+ a_1 \varepsilon^* \tag{53}$$

and

$$\lambda^\# = A_1'^+ T_1 - A_1'^+ B_1 \lambda^* \tag{54}$$

$$\lambda^* = B_1^+ T_1 - B_1^+ A_1' \lambda^\#. \tag{55}$$

In order for the filtered wrench and twist to be the same before and after the collineation, it is necessary and sufficient that they are formed as the same linear combination of the corresponding wrench m-systems and twist n-systems. That is, ε^* and λ^* must be invariants and

$$w_1^* = a_1 \varepsilon^* = \hat{K} a \varepsilon^* = \hat{K} w^* \tag{56}$$

$$T_1^* = B_1 \lambda^* = \hat{k} B \lambda^* = \hat{k} T^* \tag{57}$$

which also indicates that w^* and T^* must themselves transform under a collineation in the same manner as any other wrench and twist.

From (29) and (30), in order for ε^* and λ^* to be invariants it is necessary for the last term in (52) and (55) to vanish. The first manner in which they may happen is

$$\varepsilon^{\#} = 0 \quad \text{and} \quad \lambda^{\#} = 0. \tag{58}$$

This means the wrench w and twist T were in fact originally specified to be consistent with the environment a and B. The filtering does not alter this relationship and this case is not considered further. The second manner for ε^* and λ^* to be invariant is for the vanishing of

$$a_1^+ b_1' = (a^T(\hat{K}^T \tilde{\Gamma}_1 \hat{K})a)^{-1} a^T(\hat{K} \tilde{\Gamma}_1 \hat{K})b' = 0 \tag{59}$$

$$B_1^+ A_1' = (B^T(\hat{k} \tilde{\gamma}_1 \hat{k})B)^{-1} (\hat{k}^T \tilde{\gamma}_1 \hat{k})A' = 0. \tag{60}$$

In order that these relations be independent of any specific systems a, B and be merely a function of the collineation, then to a scalar multiple it is necessary and sufficient that

$$\hat{K}^T \tilde{\Gamma}_1 \hat{K} = \tilde{\Gamma}_1 \quad \text{and} \quad \hat{k}^T \tilde{\gamma}_1 \hat{k} = \tilde{\gamma}_1 \tag{61}$$

for which (59), (60) become

$$a_1^+ b_1' = a^+ b' = (a^T \tilde{\Gamma}_1 a)^{-1} a^T \tilde{\Gamma}_1 b' = 0 \tag{62}$$

$$B_1^+ A_1' = B^+ A' = (B^T \tilde{\gamma}_1 B)^{-1} B^T \tilde{\gamma}_1 A' = 0 \tag{63}$$

which clearly vanish by (17), (18).

Therefore, in order that invariant results are yielded by "orthogonal" projection and consequently by the "orthogonal" left pseudoinverse and by "orthogonal" projectors, it is in general necessary that all permissible collineations are restricted to those which satisfy (61). However from Section 3.2, such collineations have already been identified as elliptic collineations and were referred to as elliptic-orthogonal since

$$\hat{K}^{-1} = \hat{K}^{T} \quad , \quad \hat{k}^{-1} = \hat{k}^{T}. \tag{64}$$

Rigid body Euclidean collineations are classified as either rotations or translations and in Section 4.1 it was shown that Euclidean rotations form a subgroup of elliptic collineations and thus satisfy (61). In contrast, Euclidean translations do not have this property and thus "orthogonal" projection yields noninvariant results with respect to this group of collineations. This is illustrated in the following section by a simple numerical example.

As previously mentioned, "orthogonal" projection is also noninvariant with respect to changes in unit length. It should be recalled from Section 3.4, that the first and last three components of a wrench represent force and moment respectively. Thus only the latter three components of a wrench or a screw in ray coordinates are altered by a change in unit length. For a twist, the first three and last three coordinates represent translational and angular velocity respectively. Hence only the first three coordinates of twist or a screw in axis coordinates are altered by a change in unit length. This problem may be most easily treated by employing the previous notation and analysis,

$$w_1 = \hat{K}\, w \quad , \quad T_1 = \hat{k}\, T \tag{65}$$

where now,

$$\hat{K} = \begin{bmatrix} I_3 & \cdot \\ \cdot & \delta I_3 \end{bmatrix} , \quad \hat{k} = \begin{bmatrix} \delta I_3 & \cdot \\ \cdot & I_3 \end{bmatrix} \tag{66}$$

and

$$\hat{k} = \delta^3\, \hat{K}^{-T} . \tag{67}$$

The quantity δ represents a conversion factor such as 12 in/ft. It is easily verified that (66) satisfies the tetrahedron relationship of Section 2.3 and thus represents a projective collineation. However, from the previous analysis it has already been shown that the only collineations for which "orthogonal" projections remains invariant are elliptic collineations. Clearly, (66) does not satisfy the conditions (64) and thus is not an elliptic collineation. Therefore, "orthogonal" projections are noninvariant under changes in unit length.

To describe one further point it is useful to introduce the vector formulations of w and T given in (3.4.61) and (3.4.66),

$$w = \begin{bmatrix} \underline{f} \\ \underline{m}_0 \end{bmatrix} , \quad T = \begin{bmatrix} \underline{V}_0 \\ \underline{\Omega} \end{bmatrix} . \tag{68}$$

The pitch and perpendicular vector from the origin to the screw axis is given for the wrench by (see Section 3.4)

$$h_w = \frac{\underline{f} \cdot \underline{m}_0}{\underline{f} \cdot \underline{f}} , \quad \underline{r}_w = \frac{\underline{f} \times \underline{m}_0}{\underline{f} \cdot \underline{f}} \tag{69}$$

and for the twist by

$$h_T = \frac{\underline{\Omega} \cdot \underline{V}_0}{\underline{\Omega} \cdot \underline{\Omega}} , \quad \underline{r}_T = \frac{\underline{\omega} \times \underline{V}_0}{\underline{\Omega} \cdot \underline{\Omega}} . \tag{70}$$

After a change in unit length the virtual work is given by

$$T_1^T w_1 = \delta \, T^T w \tag{71}$$

and the pitches and perpendicular vectors become

$$h_{w1} = \delta \, h_w , \quad \underline{r}_{w1} = \delta \, \underline{r}_w \tag{72}$$

$$h_{T1} = \delta \, h_T , \quad \underline{r}_{T1} = \delta \, \underline{r}_T . \tag{73}$$

Thus the change in unit length multiples the virtual work, the pitches and the perpendicular vectors all by the conversion factor δ which is correct since these quantities are linear in the dimension of length.

It is now useful to examine the introduction of a scalar term ρ in Mason [1981] which however is not found in Mason [1978]. Wrenches and twists are both described as "points" in a six-space after they have been altered to the forms

$$w_\rho = \begin{bmatrix} \underline{f} \\ \underline{m}_0/\rho \end{bmatrix} , \quad T_\rho = \begin{bmatrix} \underline{V} \\ \rho\underline{\Omega} \end{bmatrix} \tag{74}$$

where the scalar ρ is introduced and which has dimensions of length. No clear explanation of ρ is given except to indicate that it is necessary for the construction of "orthogonal" spaces. No explanation is attempted here but a few properties are examined. All the components of w_ρ have the dimensions of force whereas for T_ρ they are all length/second. The virtual work is given by

$$T_\rho^T w_\rho = T^T w \tag{75}$$

which remains unchanged. However the pitches and perpendicular vectors are

$$h_{w\rho} = \frac{h_w}{\rho} , \quad \underline{r}_{w\rho} = \frac{\underline{r}_w}{\rho} \tag{76}$$

$$h_{T\rho} = \frac{h_T}{\rho} , \quad \underline{r}_{T\rho} = \frac{\underline{r}_T}{\rho} . \tag{77}$$

Thus by the same factor the pitches and perpendicular vectors are altered which clearly changes the physical properties of the screws. Further, the magnitudes of w and T (see Sections 3.3 and 3.4)

$$\| w \| = \| \underline{f} \| \quad , \quad \| T \| = \| \underline{\Omega} \| \tag{78}$$

become

$$\| w_\rho \| = \| w \| \quad , \quad \| T_\rho \| = \rho^2 \| T \| \; . \tag{79}$$

The only property of both wrenches and twists that is apparently unaltered by the introduction of ρ is the direction of their axes.

Summarizing, hybrid control theory which is based on "orthogonal" projection has been shown to be noninvariant with respect to Euclidean translations and changes in unit length. Thus the noninvariant theory is not based on well-defined geometric concepts and its value must be highly questionable. In the following section, further noninvariant properties of hybrid control are examined.

Section 5.3 Noninvariant Properties

> Mr. Cartesian was very unhappy. The generality of the subject was too great for his comprehension. He had an attachment to the x, y, z, which he regarded as the ne plus ultra of dynamics. 'Why will you burden the science,' he sighs, 'with all these additional names? Can you not express what you want without talking about cylindroids, and twists, and wrenches, and impulsive screws, and instantaneous screws, and all the rest of it?' 'No,' said Mr. One-to-One, 'there can be no simpler way of stating the results than the natural method we have followed. You would not object to the language if your ideas of natural phenomena had been sufficiently capacious. We are dealing with questions of perfect generality, and it would involve a sacrifice of generality were we to speak of the movement of a body except as a twist, or of a system of forces except as a wrench.'
>
> 'But,' said Mr. Commonsense, 'can you not as a concession to our ignorance tell us something in ordinary

> language which will give an idea of what you mean . . . ? Pray for once sacrifice this generality you prize so much and put the theory into some familiar shape that ordinary mortals can understand.'
>
> Mr. Anharmonic would not condescend to comply with this request, so the chairman called upon Mr. One-to-One, who somewhat ungraciously consented. 'I feel,' said he, 'the request to be an irritating one. Extreme cases frequently make bad illustrations of a general theory. That zero multiplied by infinity may be anything is not surely a felicitous exhibition of the perfections of the multiplication table. It is with reluctance that I divest the theory of its flowing geometrical habit, and present it only as a stiff conventional guy from which true grace has departed.
>
> 'Let us suppose that the rigid body, instead of being constrained as heretofore in a perfectly general manner, is subjected merely to a special type of constraint. Let it in fact be only free to rotate around a fixed point. The beautiful fabric of screws, which so elegantly expressed the latitude permitted to the body before, has now degenerated into a mere horde of lines all stuck through the point. Those varieties in the pitches of the screws which gave colour and richness to the fabric have also vanished, and the pencil of degenerate screws have a monotonous zero of pitch. Our general conceptions of mobility have thus been horribly mutilated and disfigured before they can be adapted to the old and respectable problem of the rotation of a rigid body about a fixed point. For the dynamics of this problem the wrenches assume an extreme and even monstrous type. Wrenches they still are, as wrenches they ever must be, but they are wrenches on screws of infinite pitch; they have even ceased to possess definite screws as homes of their own. We often call them couples. R.S. Ball [1887, pp. 505-506]

In the previous section, it was shown that kinestatic filtering based on "orthogonal" projections yields non-invariant results. Here "orthogonal" projection is considered in terms of minimization principles as an alternative point of view and the results are applied directly to serial manipulators. An example is presented for a five degree-of-freedom manipulator which clearly demonstrates the noninvariance of "orthogonal" projection with respect to Euclidean translations. Further noninvariant properties are discussed along with some invariant ones for redundant manipulators.

As previously shown for the noninvariant theory, a specified wrench w and twist T may be represented by the "orthogonal" decompositions

$$w = w^{*} + w^{\#} \quad , \quad T = T^{*} = T^{\#} \tag{1}$$

or equivalently by

$$w = [a \; b'] \begin{bmatrix} \varepsilon^{*} \\ \varepsilon^{\#} \end{bmatrix} \; , \quad T = [A' \quad B] \begin{bmatrix} \lambda^{\#} \\ \lambda^{*} \end{bmatrix} \tag{2}$$

where a is an m-system of wrenches, B is an n-system of twists and the "orthogonal" systems satisfy

$$a^{T} b' = 0 \quad , \quad B^{T} A' = 0. \tag{3}$$

The "orthogonal" projection method may be stated as the solution to the minimization problem

$$\text{minimize:} \quad e = \tfrac{1}{2} w^{\#T} \, \tilde{\Gamma}_{1} \, w^{\#} \tag{4}$$

where

$$\tilde{\Gamma}_{1} = \tilde{\gamma}_{1} = \tilde{I}_{6}. \tag{5}$$

Alternatively using (1) and (2), equation (4) may be expressed as

$$\text{minimize:} \quad e = \tfrac{1}{2}(w - a\varepsilon^{*})^{T} \, \tilde{\Gamma}_{1}(w - a\varepsilon^{*}) \tag{6}$$

which is often referred to as the least squares problem. Setting the partials to zero,

$$\frac{\partial e}{\partial \varepsilon_{i}^{*}} = 0 \, , \quad i = 1 \, . \, . \, . \, m \tag{7}$$

and then writing the equations in matrix form leads to the well-known left pseudoinverse solution

$$\varepsilon^{*} = (a^{T} \, \tilde{\Gamma}_{1} \, a)^{-1} \, a^{T} \, \tilde{\Gamma}_{1} \, w = a^{+} \, w. \tag{8}$$

The filtered wrench may be determined using an "orthogonal" projector matrix,

$$w^{*} = aa^{+} = P_{a} \, w. \tag{9}$$

Analogously, the least squares minimization problem for twists is

$$\text{minimize:} \quad E = \tfrac{1}{2}T^{\#T} \, \tilde{\gamma}_{1} \, T^{\#} \tag{10}$$

or equivalently

$$\text{minimize:} \quad E = \tfrac{1}{2}(T - B\lambda^{*}) \quad \tilde{\gamma}_{1}(T - B\lambda^{*}) \tag{11}$$

for which the solution is

$$\lambda^{*} = (B^{T} \, \tilde{\gamma}_{1} \, B)^{-1} \, B^{T} \, \tilde{\gamma}_{1} \, T = B^{+} \, T \tag{12}$$

and

$$T^{\star} = BB^{+}\, T = p_B\, T. \tag{13}$$

It should be noted that the solutions (8), (9) and (12), (13) are identical to the ones obtained in Section 5.2 by "orthogonal" projection. Further, it should clear that the quadratic forms e and E in (6) and (10) are noninvariant with respect to general Euclidean translations and changes in unit length. However on reflection, such minimization problems are illogical since the forms e and E are devoid of any geometrical meaning. For example, expressing the "error" wrench and twist in a vector formulation,

$$w^{\#} = \begin{bmatrix} \underline{f}^{\#} \\ \underline{m}_0^{\#} \end{bmatrix}, \qquad T^{\#} = \begin{bmatrix} \underline{v}_0^{\#} \\ \underline{\Omega}^{\#} \end{bmatrix} \tag{14}$$

and using (5) the quadratic forms may be expressed as

$$2e = (\underline{f}^{\#})^2 + (\underline{m}_0^{\#})^2\ , \ 2E = (\underline{v}_0^{\#})^2 + (\underline{\Omega}^{\#})^2. \tag{15}$$

No attempt is made here to interpret such expressions.

However, there is another expression that under special circumstances may be confused with E in (10) or (11). Let $\tilde{\Pi}$ be the inertia dyadic at the point on a rigid body which is coincident with the origin of coordinates. A twist T is specified for the body whose motion is constrained to be in the n-system B. A possible filtered twist $T^{\star}$ may be determined by minimizing the kinetic energy of an "error" twist

$$\text{minimize:} \quad KE = \tfrac{1}{2}(T - B\lambda^{*})^{T}\,\tilde{\Pi}\,(T - B\lambda^{*}) \qquad (16)$$

the solution of which is simply

$$\lambda^{*} = (B^{T}\,\tilde{\Pi}\,B)^{-1}\,B^{T}\,\tilde{\Pi}\,T. \qquad (17)$$

Since kinetic energy is an invariant quantity it is clear that both (16) and (17) are invariants. Under a Euclidean collineation the twists and the inertia dyadic transform by

$$T_1 = \hat{k}\,T, \; B_1 = \hat{k}\,B \qquad (18)$$

and

$$\tilde{\Pi}_1 = k^{-T}\,\tilde{\Pi}\,k^{-1}. \qquad (19)$$

Using (18) and (19) in a transformed expression analogous to (16) yields back the same result (17) as expected.

Consider now that the origin is located at the center of mass of the rigid body and that the principal axes of inertia are along the coordinate axes. Further, suppose that the mass and the principal inertias of the body all have unity magnitude. Under these conditions (16) may be expressed as

$$\text{minimize:} \quad KE = \tfrac{1}{2}(T - B\lambda^{*})^{T}\,\tilde{I}_6\,(T - B\lambda^{*}) \qquad (20)$$

which is numerically identical to (11) using $\tilde{\gamma}_1 = \tilde{I}_6$ in (5). However, there is a very important distinction between the two expressions. The nature of "orthogonal" projection requires that $\tilde{\gamma}_1$ has the same form regardless of origin location which is responsible for the noninvariance whereas the inertia dyadic $\tilde{\Pi}$ transforms by (19) which is necessary for the invariance of kinetic energy.

Thus far in this section and in the previous one "orthogonal" projection and filtering has been applied to a manipulator workpiece which has been modeled as a rigid body. The analysis is easily extended to include serial manipulators with n degrees-of-freedom where $1 \leq n \leq 5$. First the noninvariant twist solution is illustrated by a numerical example and is then followed by a noninvariant wrench analysis by introducing the right pseudoinverse.

The instantaneous twist T of the end-effector for an n-jointed serial manipulator is given by,

$$T = B\ \omega \qquad (21)$$

where B is now the $6 \times n$ Jacobian whose columns specify the n unit screw vectors of the joints in axis coordinates and ω is an $n \times 1$ array which specifies the angular (or translational) speeds of the joints. (It should be noted that in Section 5.1 the Jacobian was designated by J whereas here B is employed in order to facilitate using the previous analysis of the workpiece.) In general, it is assumed that the specified twist T is not in the column space of B and the "orthogonal" projection solution yields

$$\omega^{*} = B^{+} T \quad , \quad T^{*} = p_B T. \tag{22}$$

A simple example illustrates the noninvariance of the "orthogonal" projection solution (22). Figure 5.3.1 shows a five axis Cartesian manipulator where the first three joints are prismatic and at right angles and the last two joints are revolute and cointersect at a point coincident with the origin 0. The coordinate directions are indicated by the unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$. For origin 0 the manipulator Jacobian is given by

$$B = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \underline{i} & \underline{j} \end{bmatrix} . \tag{23}$$

The single wrench reciprocal to B and the single twist "orthogonal" to B are specified respectively by

$$a = \begin{bmatrix} \cdot \\ \underline{k} \end{bmatrix} , \quad A' = \begin{bmatrix} \cdot \\ \underline{k} \end{bmatrix} \tag{24}$$

where a is a couple in the $\underline{k}$ direction and A' is a rotation in the $\underline{k}$ direction through 0.

(As an aside, it is useful to note that in (23) B may also represent five linearly independent joints of a six degree-of-freedom Cartesian manipulator which is in a singularity configuration. The sixth joint is a revolute through 0 and for the singular case at hand it is coplanar with joints four and five or more particularly collinear with joint four. Thus the following analysis also applies

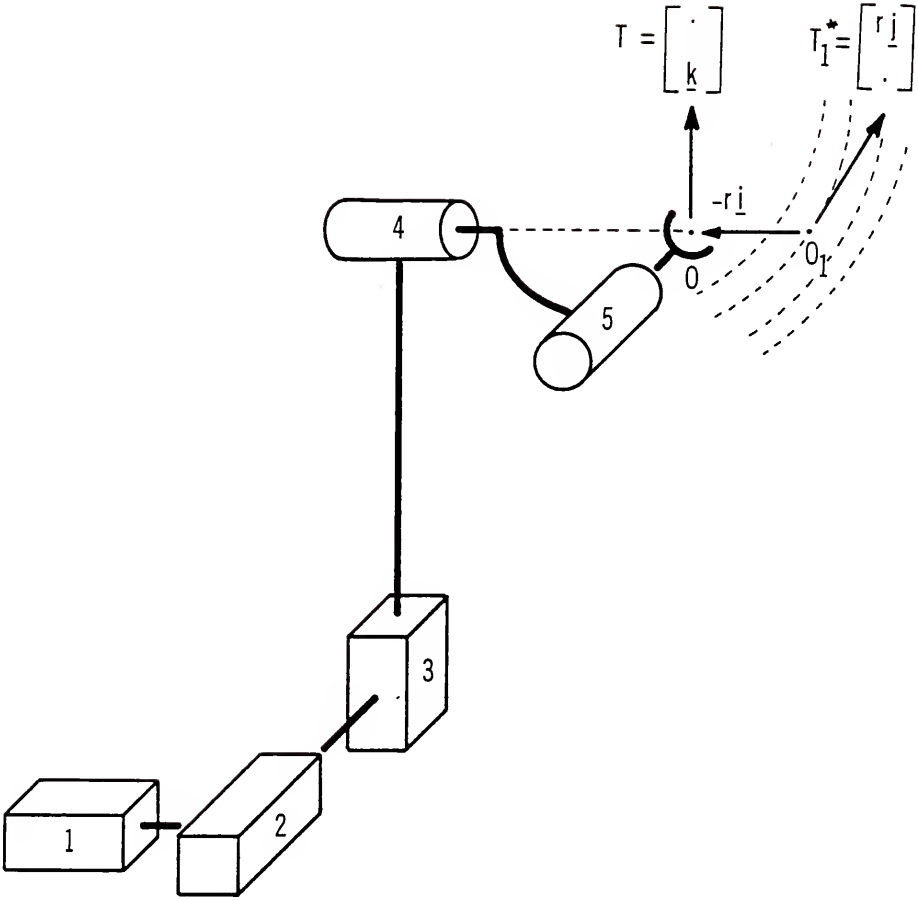

Figure 5.3.1 A five axis Cartesian manipulator.

to using "orthogonal" projection in connection with singular manipulator configurations.)

For the most transparent example, let the desired twist be in the "orthogonal" space,

$$T = \begin{bmatrix} \cdot \\ \underline{k} \end{bmatrix} = A' \ . \qquad (25)$$

Solving for the joint speeds

$$\omega^* = B^+ T = B^+ T = 0 \qquad (26)$$

and the filtered twist

$$T^* = p_B T = BB^+ T = 0 \qquad (27)$$

then it is clear that all actuators are fixed and consequently the end-effector is stationary.

It is next necessary to demonstrate that the results of "orthogonal" projection differ after a translation of space $\hat{k}$. However, a collineation of space may be equivalently represented as a transformation of the origin by $\hat{k}^{-1}$ and may be used to simplify the problem. Let the origin 0 be translated r units in the $\underline{i}$ direction to 0_1 where in Fig. 5.3.1 the vector $-r\underline{i}$ is shown. From Section 3.3, the equivalent translation of space is given by,

$$\hat{k} = \begin{bmatrix} I_3 & -ri^* \\ \cdot & I_3 \end{bmatrix} \qquad (28)$$

where the skew-symmetric matrix $-i^*$ is defined by

$$-i^* \, \underline{y} = -\underline{i} \times \underline{y} \tag{29}$$

for all vectors $\underline{y}$.

In terms of origin 0_1 the Jacobian is now given by

$$B_1 = \hat{k}_1 \, B = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} & \cdot & -r\underline{k} \\ \cdot & \cdot & \cdot & \underline{i} & \underline{j} \end{bmatrix} \tag{30}$$

and the specified twist is

$$T_1 = \hat{k} \; T = \begin{bmatrix} r\,\underline{j} \\ \underline{k} \end{bmatrix} . \tag{31}$$

For completeness, the remaining calculations are detailed:

$$B_1^T \, B_1 = \begin{bmatrix} I_2 & \cdot & \cdot & \cdot \\ \cdot & 1 & 0 & -r \\ \cdot & 0 & 1 & 0 \\ \cdot & -r & 0 & r^2+1 \end{bmatrix} \tag{32}$$

$$(B_1^T \, B_1)^{-1} = \begin{bmatrix} I_2 & \cdot & \cdot & \cdot \\ \cdot & r^2+1 & 0 & r \\ \cdot & 0 & 1 & 0 \\ \cdot & r & 0 & 1 \end{bmatrix} \tag{33}$$

$$B_1^+ = (B_1^T \, B_1)^{-1} \, B_1^T = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} & \cdot & \cdot \\ \cdot & \cdot & r\underline{j} & \underline{i} & \underline{j} \end{bmatrix}^T \tag{30}$$

$$p_{B1} = B_1 \, B_1^{+} = \begin{bmatrix} I_5 & \cdot \\ \cdot & \cdot \end{bmatrix} \tag{31}$$

$$\omega_1^{*} = B_1^{+} \, T_1 = [0 \quad r \quad 0 \quad 0 \quad 0] \tag{32}$$

$$T_1^{*} = p_{B1} \, T_1 = \begin{bmatrix} r\underline{j} \\ \cdot \end{bmatrix} . \tag{33}$$

Therefore with respect to origin 0_1, the second actuator velocity in (32) is r whereas it was previously zero and from (33) the end-effector is translating with velocity $r\underline{j}$ where previously it was stationary. The results of this simple example clearly illustrate the noninvariance of "orthogonal" projection with respect to translations.

Because this example is so trivial, it admits a further explanation. First consider that a moving space (an extended rigid body) is rotating under the twist T through 0, see Fig. 5.3.1. Each point in the moving space has an instantaneous tangential velocity and the totality of such form a cylindrical velocity field. In particular, at a point coincident with 0_1 the velocity is equivalent to the filtered twist T_1^*. Using a result illustrated in Fig. 4.4.4, displacing 0_1 through a finite rotation about the axis of T, merely displaces the filtered twist by the same rotation. Further, a translation of 0_1 parallel to the axis of T in the $\underline{k}$ direction does not alter T_1^*. Thus for any location of 0_1, the filtered twist T_1^* is a pure translational velocity which is numerically equivalent to the velocity of a point generated by the twist T. Since there are ∞^2 different velocities in a cylindrical field, it follows that there are ∞^2 possible solutions to this problem using "orthogonal" projection.

For general n-systems, the resulting filtered twists are a complicated function of origin location for any particular specified twist. It should be noted that although "orthogonal" projection yields noninvariant results they are at least systematic and for any particular origin the solution is unique. However, it is also true that any twist selected randomly from the n-system may also be used as a filtered twist. Clearly such a random filter is undesirable since for a continuous variation of the specified twist the filtered twists will vary discontinuously. Apparently "orthogonal" projection will yield a continuous set of filtered twists and thus has definite advantages over random selection. However the noninvariance of "orthogonal" projection makes it an illogical choice for practical and theoretical use. New methods are necessary that preserve both continuity and invariant properties and are discussed in Section 5.4.

In connection with changes of origin location there is a simple concept that should be clearly understood. Rather than formulating the Jacobian based on a particular origin, a so-called point of interest on the end-effector is sometimes used. The Jacobian relates the joint velocities to the instantaneous motion of the end-effector characterized at the point of interest. Thus the Jacobian is independent of the origin and it may then be argued that "orthogonal" projection is invariant with respect to origin translations. While this is true, "orthogonal" projection is clearly not independent of a change in the point of interest. The reason for any apparent discrepancy is that when using Plucker coordinates the origin and the so-called point of interest are always coincident.

Frequently a set of coordinate directions are associated with a point of interest as its origin. The frame may be fixed or moving depending on the details of the analysis. Sometimes this is referred to as a constraint frame and a particular selection may enable the simplest specification of the constraint and freedom of a rigid body. Clearly the constraint and freedom of a body are independent of any type of constraint frame and although parameterizations may differ, physical and geometrical properties must remain invariant. "Orthogonal" projection is however noninvariant with translations of the constraint frame.

Returning to the analysis of a general n degree-of-freedom manipulator where $1 \leq n \leq 5$, it is useful to briefly examine the application of the "orthogonal" right pseudoinverse to wrench determination. By the principle of virtual work it has previously been shown that for a wrench w applied to the manipulator end-effector, the actuator torques τ that are necessary to maintain static equilibrium are given by

$$\tau = B^T w. \tag{34}$$

In general it is clear that there are an infinite number of such wrenches that produce the same torque values since

$$B^T(w + a\mu) = B^T w = \tau \tag{35}$$

where a is the m-system reciprocal to B and μ is an $m \times 1$ array of arbitrary multipliers. The right pseudoinverse may be used to determine a unique wrench w^* which is the solution to the minimum "norm" problem,

$$\text{minimize:} \quad \tfrac{1}{2} w^{*T} \tilde{\Gamma}_1 w^* \tag{36}$$

$$\text{subject to:} \quad B^T w^* - \tau = 0 \tag{37}$$

where $\tilde{\Gamma}_1 = \tilde{\gamma}_1 = \tilde{I}_6$. It should be noted that the meaning of w^* here is distinct from that in Section 5.2. By applying Lagrangian multipliers the solution is given by

$$w^* = \tilde{\gamma}_1 B (B^T \tilde{\gamma}_1 B)^{-1} \tau. \tag{38}$$

Under a collineation of space,

$$w_1^* = \hat{k} w , \quad B_1 = \hat{K} B , \quad \hat{k} = \hat{K}^{-T} \tag{39}$$

(38) becomes

$$w_1^* = \tilde{\gamma}_1 B_1 (B_1^T \tilde{\gamma}_1 B_1)^{-1} \tau \tag{40}$$

and using (39) yields

$$w^{*} = (K^{T}\ \tilde{\gamma}_{1}\ \hat{K})\ B(B^{T}(\hat{K}^{T}\ \tilde{\gamma}_{1}\hat{K})\ B)^{-1}\ \tau. \tag{41}$$

By previous arguments in Section 5.2, for w^{*} to be independent of any particular n-system B, then to a scalar multiple, $\hat{K}$ must be an elliptic collineation. Thus the right pseudoinverse applied to this problem yields a result which is noninvariant with respect to Euclidean translations and changes in unit length.

For the purpose of symmetry and completeness it is useful to briefly examine the application of pseudoinverses to redundant manipulators where $n > 6$. The twist equation

$$T = B\ \omega \tag{42}$$

where B is of full row rank, may be solved using a weighted minimum norm formulation,

$$\text{minimize:}\quad \tfrac{1}{2}\omega^{*T}\ \tilde{\pi}\ \omega^{*} \tag{43}$$

$$\text{subject to:}\quad B\ \omega^{*} - T = 0. \tag{44}$$

In (43), $\tilde{\pi}$ is a general positive definite weighting matrix. An application of Lagrangian multipliers yields a right pseudoinverse solution,

$$\omega^{*} = \tilde{\pi}^{-1}\ B^{T}(B\ \tilde{\pi}^{-1}\ B^{T})^{-1}\ T \tag{45}$$

which is discussed in Whitney [1972]. It should be noted that (45) not only remains invariant under any collineation of space but also under any homographic transformation k where

$$T_1 = k\,T\,, \quad B_1 = k\,B. \tag{46}$$

This may be easily verified by rewriting (45) using T_1, B_1 and then employing (46).

For a redundant manipulator in static equilibrium the joint torques are given by

$$\tau = B^T w \tag{47}$$

where in general B^T is of full column rank. It is possible to solve (47) using a weighted least squares formulation,

$$\text{minimize:} \quad \tfrac{1}{2}(\tau - B^T w^*)^T \,\tilde{\Pi}(\tau - B^T w^*) \tag{48}$$

where $\tilde{\Pi}$ is a positive definite weighting matrix. The solution to (48) is given by a left pseudoinverse

$$w^* = (B\,\tilde{\Pi}\,B^T)^{-1}\,B\,\tilde{\Pi}\,\tau. \tag{49}$$

Analogous to (45), it may be easily shown that (49) is invariant with respect to a general homographic transformation $k = K^{-T}$.

For the case of a six degree-of-freedom manipulator, the left and right pseudoinverses are the same for a nonsingular B of rank n = 6,

$$\omega = B^{-1}\,T\,, \qquad \tau = B^{-T}\,w. \tag{50}$$

Under a general homographic transformation of space $k = K^{-T}$,

$$\omega = (k\ B)^{-1}\ (k\ T), \quad \tau = (k\ B)^{-T}\ (K\ w) \qquad (51)$$

for which ω and τ clearly remain invariant.

There are many types of generalized inverses other than pseudoinverses. However by the preceding analyses it should be clear that it is necessary for them to yield results which are invariant with respect to Euclidean collineations in order to be applicable in general mechanics. For example, the (1, 2) generalized inverse $B^\dagger$ is defined in Ben-Israel and Greville [1974] as a matrix satisfying the two conditions

$$B\ B^\dagger\ B = B \qquad (52)$$

$$B^\dagger\ B\ B^\dagger = B^\dagger. \qquad (53)$$

Under the collineation $B_1 = \hat{k}\ B$ the conditions become,

$$(\hat{k}\ B)\ B_1^\dagger (\hat{k}\ B) = (\hat{k}\ B)' \qquad (54)$$

$$B_1^\dagger (\hat{k}\ B) B_1^\dagger = B_1^\dagger. \qquad (55)$$

From (54) it is easily deduced that for invariance it is necessary that the transformation of the (1, 2) inverse is

$$B_1^\dagger = B^\dagger\ \hat{k}^{-1} \qquad (56)$$

and since this relation additionally satisfies (55) it is also sufficient. The method of computing $B^\dagger$ is not given here but may be found in Ben-Israel and Greville [1974,

p. 284] or in West and Asada [1985] where it is used in a hybrid control formulation. Without presenting the details here, whether or not $B^{\dagger}$ is invariant depends on the structure of B. For example, if B is of full row rank where $n > 6$ then the (1, 2) inverse is generally invariant (assuming that the permutation matrices used in the solution are unity). If B is of full column rank and $n < 6$ then in general the (1, 2) inverse is not invariant with respect to Euclidean collineations (including rotations) which is most easily demonstrated by an example. These results parallel the previous ones with regard to the right pseudoinverse of B for $n > 6$ being invariant and the left pseudoinverse of B for $n < 6$ being noninvariant. However, to apply the (1, 2) inverse B does not have to be of full row or column rank. This must be considered for a complete characterization of invariance which is not attempted here.

Finally, it is interesting to directly examine the "orthogonality" of screws with respect to Euclidean translations. Using ray coordinates, two screws (or wrenches) p and q may be expressed in a vector formulation by

$$p = \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} \qquad q = \begin{bmatrix} \underline{q} \\ \underline{q}_0 \end{bmatrix}. \tag{57}$$

In general p and q are not "orthogonal" and it is desired to determine the locus of origin locations where they become "orthogonal." A translation of the origin by $\underline{t}$ is represented as the collineation

$$\hat{k} = \begin{bmatrix} I_3 & \cdot \\ -t^* & I_3 \end{bmatrix} \qquad (58)$$

and the "orthogonality" condition is

$$p^T \, \hat{k}^T \, \tilde{\Gamma}_1 \, \hat{k} \, q = 0 \qquad (59)$$

Using (58) and $\tilde{\Gamma}_1 = \tilde{I}_6$ in (59) yields

$$p^T \begin{bmatrix} I_3 + t^{*T} t^* & t^* \\ t^{*T} & I_3 \end{bmatrix} q = 0. \qquad (60)$$

Substituting the vector expressions (57) in (60) gives after some manipulation

$$t^T \, \tilde{\Psi}_1 \, t = 0 \qquad (61)$$

where

$$\tilde{\Psi}_1 = \begin{bmatrix} s^T \, \tilde{\Gamma}_1 \, p & (\underline{s}_0 \times \underline{p})^T \\ (\underline{p}_0 \times \underline{s}) & \underline{s}^T \, \underline{p} \, I_3 - \underline{s} \, \underline{p}^T \end{bmatrix}, \quad t = \begin{bmatrix} 1 \\ \underline{t} \end{bmatrix}. \qquad (62)$$

Since (61) is a quadratic form, it may be expressed in terms of a symmetrical matrix,

$$t^T \, \tilde{\Psi} \, t = 0 \qquad (63)$$

where

$$\tilde{\Psi} = \tfrac{1}{2}(\tilde{\Psi}_1 + \tilde{\Psi}_1^T). \qquad (64)$$

Thus from (63), the locus of points t which specifies the origin locations where the screws p and q become "orthogonal" is a general quadric surface.

Section 5.4 Invariant Formulations

> The final meeting showed that real dynamical enthusiasm had been kindled in the committee. Vistas of great mathematical theories were opened out in many directions. One member showed how the theory of screws could be applied not merely to a single rigid body but to any mechanical system whatever. He sketched a geometrical conception of what he was pleased to call a screw-chain, by which he said he could so bind even the most elaborate system of rigid bodies that they would be compelled to conform to the theory of screws. Nay, soaring still further into the empyrean, he showed that all the instantaneous motions of every molecule in the universe were only a twist about one screw-chain while all the forces of the universe were but a wrench upon another.
>
> Mr. One-to-One expounded the "Ausdehnungslehre' and showed that the theory of screws was closely related to parts of Grassmann's great work; while Mr. Anharmonic told how Sir W. R. Hamilton, in his celebrated "Theory of systems of rays" had by his discovery of the cylindroid helped to lay the foundation of the Theory of Screws.
>
> The climax of mathematical eloquence was attained in the speech of Mr. Querulous, who, with newborn enthusiasm, launched into appalling speculations. He had evidently been reading his 'Cayley' and had become conscious of the poverty of geometrical conception arising from our unfortunate residence in a space of an arbitrary and unsymmetrical description.
>
> 'Three dimensions,' he said, 'may perhaps be enough for an intelligent geometer. He may get on fairly well without a four-dimensioned space, but he does most heartily remonstrate against a flat infinity. Think of infinity,' he cries, 'as it should be, perhaps even as it is. Talk not of your scanty straight line at infinity and your miserable pair of circular points. Boldly assert that infinity is an ample quadric, and not the mere ghost of one; and then geometry will become what geometry ought to be. Then will every

> twist resolve into a right vector and a left vector, as the genius of Clifford proved. Then will the theory of screws shed away some few adhering incongruities and fully develop its shapely proportions. Then will--' But here the chairman said he feared the discussion was beginning to wax somewhat transcendental. For his part he was content with the results of the experiments even though they had been conducted in the vapid old space of Euclid. He reminded them that their function had now concluded, for they had ascertained everything relating to the rigid body which had been committed to them. He hoped they would agree with him that the enquiry had been an instructive one. They had been engaged in the study of Nature, they had approached the problems in the true philosophical spirit, and the rewards they had obtained proved that
>
> 'Nature never did betray
> The heart that truly loved her.'
>
> R.S. Ball [1887, p. 509]

In the previous sections, it has been shown that using "orthogonal" projection for kinestatic filtering leads to noninvariant results with respect to Euclidean translations and changes in the unit of length. In this section, methods of invariant kinestatic filtering are presented along with identifying the areas of their applicability. The formulations are developed using methods of general projection and unconstrained classical optimization. The development demonstrates that invariant kinestatic filtering does not have a unique solution but may vary according to the criteria which are utilized. Hopefully the initial attempts presented here will provide a foundation for further developments.

As in the previous sections, the constrained motion of a workpiece is considered where w is a specified reaction wrench and T is a specified twist. When w and T are consistent with the environment then the freedom equations of the body are

$$w = a\,\varepsilon\,, \quad T = B\,\lambda \tag{1}$$

and the constraint equations of the body are

$$B^T w = 0, \quad a^T T = 0 \tag{2}$$

where a is an m-system of screws in ray coordinates, B is an n-system of screws in axis coordinates and

$$m = n - 6. \tag{3}$$

The systems a and B are reciprocal for which

$$B^T a = 0 \tag{4}$$

and

$$T^T w = 0. \tag{5}$$

The m columns of a and the n columns of B are said to each form a basis for their respective systems. A basis for a system is clearly not unique. Any m linearly independent screws in the m-system and any n linearly screws in the n-system may respectively be chosen as a basis. For example, (1) may be expressed in terms of alternative bases

$$w = \bar{a}\;\bar{\varepsilon}, \quad T = \bar{B}\;\bar{\lambda} \tag{6}$$

where the new bases are given by

$$\bar{a} = a\,\alpha\ , \quad \bar{B} = B\,\beta \tag{7}$$

and the new multipliers are

$$\bar{\varepsilon} = \alpha^{-1}\,\varepsilon, \quad \bar{\lambda} = \beta^{-1}\,\lambda \tag{8}$$

where the nonsingular matrices α and β have the dimensions $m \times m$ and $n \times n$.

Three necessary conditions are now proposed for invariant kinestatic filtering:

i) Invariance with respect to rigid body Euclidean collineations of space (or coordinate frame)

ii) Invariance with respect to changes in unit length

iii) Invariance with respect to a change in basis.

The first two conditions have been discussed at length in the previous sections. The third condition may be interpreted as that the freedom and constraint of a rigid body is independent of any particular representation of the m-system a and the n-system B. That is, the columns of a, B and $\bar{a}$, $\bar{B}$ must span the same respective spaces. It should be noted that although "orthogonal" projection does not satisfy the first two conditions it does satisfy the third. Only formulations which satisfy all three conditions are referred to here as invariant.

The first formulation is presented in a manner analogous to the development in Section 5.2. However instead of using the noninvariant concept of an "orthogonal" space, the invariant concept of a reciprocal space is employed. This is most easily accomplished by introducing the transformation between ray and axis coordinates $\tilde{\Delta}$ which has been extensively detailed in the previous chapters. Thus the n-system reciprocal to a is

$$b = \tilde{\Delta}\ B \tag{9}$$

and the m-system reciprocal to B is

$$A = \tilde{\Delta}\ a \tag{10}$$

where

$$\tilde{\Delta} = \begin{bmatrix} \cdot & I_3 \\ I_3 & \cdot \end{bmatrix}. \tag{11}$$

From (4) and (9) - (11), the reciprocity condition in either ray or axis coordinates is given by

$$a^T\ \tilde{\Delta}\ b = 0 \tag{12}$$

$$A^T\ \tilde{\Delta}\ B = 0. \tag{13}$$

It is necessary to consider two distinct cases. For the first case it is assumed that the reciprocal m-system and n-system together span a six-space

$$u = [a\ b] \tag{14}$$

$$U = [A\ B]. \tag{15}$$

Any specified wrench w and twist T which in general are not in the respective systems a and B, may be expressed in terms of a $\tilde{\Delta}$-decomposition

$$w = u\ \varepsilon = [a\ b] \begin{bmatrix} \varepsilon^{*} \\ \varepsilon^{\#} \end{bmatrix} \tag{16}$$

$$T = U\ \lambda = [A\ B] \begin{bmatrix} \lambda^{\#} \\ \lambda^{*} \end{bmatrix} \tag{17}$$

where ε^{*}, $\lambda^{\#}$ are $m \times 1$ and $\varepsilon^{\#}$, λ^{*} are $n \times 1$. Forming the $\tilde{\Delta}$ inner products of (14), (15) with (16), (17) yields

$$u^{T}\ \tilde{\Delta}\ w = u^{T}\ \tilde{\Delta}\ u\ \varepsilon \tag{18}$$

$$U^{T}\ \tilde{\Delta}\ T = U^{T}\ \tilde{\Delta}\ U\ \lambda \tag{19}$$

or equivalently

$$\begin{bmatrix} a^{T}\ \tilde{\Delta}\ w \\ b^{T}\ \tilde{\Delta}\ w \end{bmatrix} = \begin{bmatrix} a^{T}\ \tilde{\Delta}\ a & . \\ . & b^{T}\ \tilde{\Delta}\ b \end{bmatrix} \begin{bmatrix} \varepsilon^{*} \\ \varepsilon^{\#} \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} A^{T}\ \tilde{\Delta}\ T \\ B^{T}\ \tilde{\Delta}\ T \end{bmatrix} = \begin{bmatrix} A^{T}\ \tilde{\Delta}\ A & . \\ . & B^{T}\ \tilde{\Delta}\ B \end{bmatrix} \begin{bmatrix} \lambda^{\#} \\ \lambda^{*} \end{bmatrix} \tag{21}$$

When the matrices in (20), (21) are invertible (this is discussed subsequently) then

$$\varepsilon^{*} = a^{+}\ w, \quad \varepsilon^{\#} = b^{+}\ w \tag{22}$$

$$\lambda^{\#} = A^{+}\ T, \quad \lambda^{*} = B^{+}\ T \tag{23}$$

where the superscript + now denotes the left $\tilde{\Delta}$-pseudoinverse which is defined for a general matrix δ by

$$\delta^{+} = (\delta^{T} \tilde{\Delta} \delta)^{-1} \delta^{T} \tilde{\Delta} \qquad (24)$$

and has the property

$$\delta^{+} \delta = I_6 . \qquad (25)$$

Substitution of (22), (23) in (16), (17) yields

$$w^{*} = P_a w , \quad w^{\#} = P_b w \qquad (26)$$

$$T^{\#} = p_A T , \quad T^{*} = p_B T \qquad (27)$$

where P and p are $\tilde{\Delta}$-projector matrices that are defined in the general form

$$\rho_{\delta} = \delta \delta^{+} \qquad (28)$$

which is idempotent

$$\rho_{\delta} \rho_{\delta} = \rho_{\delta} \qquad (29)$$

and has the same rank as δ. Further, it is easily shown that the projectors in (26), (27) have the usual properties

$$P_a + P_b = I_6 , \quad P_a P_b = 0 \qquad (30)$$

$$P_A + P_B = I_6 , \quad P_A P_B = 0. \qquad (31)$$

Equations (26), (27) represent the decompositions of w and T on the reciprocal spaces where

$$w = w^{*} + w^{\#} , \quad T = T^{\#} + T^{*} . \tag{32}$$

It is necessary to demonstrate that the results of this solution satisfy the three invariant conditions. For the first two conditions consider a collineation $\hat{k}$, $\hat{K}$. Using a general symmetrical correlation $\tilde{\Psi}, \tilde{\psi}$ then (26), (27) become

$$\hat{K} w^{*} = \hat{K} a (a^{T} (\hat{K}^{T} \tilde{\Psi} \hat{K}) a)^{-1} a^{T} (\hat{K}^{T} \tilde{\Psi} \hat{K}) w \tag{33}$$

$$\hat{k} T^{*} = \hat{k} B (B^{T} (\hat{k}^{T} \tilde{\psi} \hat{k}) B)^{-1} B^{T} (\hat{k}^{T} \tilde{\psi} \hat{k}) T \tag{34}$$

where for the case under consideration

$$\tilde{\Psi} = \tilde{\psi} = \tilde{\Delta} . \tag{35}$$

In order for (33), (34) to be invariant then it is necessary that to a scalar multiple,

$$\hat{K}^{T} \tilde{\Delta} \hat{K} = \tilde{\Delta}, \quad \hat{k}^{T} \tilde{\Delta} \hat{k} = \tilde{\Delta} \tag{36}$$

which are the tetrahedron relations discussed in Chapter 2. All induced projective collineations satisfy (36) and since rigid body Euclidean collineations form a subgroup of projective collineations the first invariant condition is satisifed.

As shown in Section 5.2, a change in scale may be effected by

$$\hat{K} = \begin{bmatrix} I_3 & \cdot \\ \cdot & \delta I_3 \end{bmatrix} , \quad \hat{k} = \begin{bmatrix} \delta I_3 & \cdot \\ \cdot & I_3 \end{bmatrix} \tag{37}$$

where δ is a conversion factor. Since (37) satisfies (36) to the scalar multiple δ, it is a projective collineation. Therefore (33), (34) are invariant under the scale change (37) and the second condition is met.

The third criteria may be demonstrated by expressing w^*, T^* in (26), (27) using the change of basis (7)

$$w^* = a\ \alpha(\alpha^T\ a^T\ \tilde{\Psi}\ a\ \alpha)^{-1}\ \alpha^T\ a^T\ \tilde{\Psi}\ w \tag{38}$$

$$T^* = B\ \beta(\beta^T\ B^T\ \tilde{\psi}\ B\ \beta)^{-1}\ \beta^T\ B^T\ \tilde{\psi}\ T. \tag{39}$$

By expanding the inverses in (38), (39), it is evident that these expressions are also invariant and thus the third condition is satisfied. It should be noted that invariance with respect to a change in basis is independent of the particular correlation $\tilde{\psi}$, $\tilde{\Psi}$. As previously mentioned, "orthogonal" projection is invariant with respect to the third condition while it is noninvariant with respect to the first two. On the other hand it should also be mentioned that formulations have been derived which satisfy the first two conditions but not the third. However, these are not presented here. Therefore the method of reciprocal or $\tilde{\Delta}$-projection will yield invariant results whenever the two reciprocal systems, the m-system a, A and the n-system b, B together space a linear six-space. This has been referred to as the first case.

The second case occurs when the two systems do not span a six-space. Let S_m and S_n represent the subspaces which correspond respectively to the m-system and n-system. Then the rank or dimension of the direct sum for the two systems is given by (see for example Birkhoff and MacLane [1965]),

$$d(S_m \dot{+} S_n) = d(S_m) + d(S_n) - d(S_m \cap S_n) \tag{40}$$

or

$$d(S_m \dot{+} S_n) = m + n - r = 6 - r \tag{41}$$

where r denotes the dimension of the intersection of S_m and S_n. Thus together a, b or A, B only span a (6-r)-system. It is useful to introduce a basis for the r-system of intersection by,

$$c = [c_1 \; . \; . \; . \; c_r] \; , \; C = [C_1 \; . \; . \; . \; C_r]. \tag{42}$$

(Frequently a screw system and its basis are referred to for brevity by the same symbol, e.g. a, B, c, etc.)

By definition, the m-system and n-system are reciprocal and since the r-system simultaneously belongs to both it follows that

$$a^T \tilde{\Delta} c = A^T \tilde{\Delta} C = 0 \tag{43}$$

$$b^T \tilde{\Delta} c = B^T \tilde{\Delta} C = 0 \tag{44}$$

and

$$c^T \tilde{\Delta} c = C^T \tilde{\Delta} C = 0. \tag{45}$$

Thus from (43) - (45) every element in the r-system is reciprocal to every element in the m, n and r-systems. The r-system is said to be $\tilde{\Delta}$-isotropic. Since each of its elements must be self-reciprocal, they represent screws of pitch 0 or ∞, that is finitely distant or infinitely distant lines.

Briefly, the reason for the existence of $\tilde{\Delta}$-isotropic subspaces is because the nonsingular matrix $\tilde{\Delta}$ is indefinite, i.e. it has three positive and negative eigenvalues. Further, since its signature is zero, it is said to form a neutral inner product space. Thorough treatments of inner product spaces and isotropic subspaces may be found in Artin [1957], Lang [1965], Kaplansky [1974], Porteous [1981]. However, it is only necessary to use a simple result from these works to determine when the projectors in (26), (27) may be formed. That is, coreciprocal or $\tilde{\Delta}$-orthogonal bases may always be selected for the m-system and the n-system.

Using (24) and (28) the general projector may be expressed by

$$\rho_\delta = \delta(\delta^T \tilde{\Delta} \delta)^{-1} \delta^T \tilde{\Delta} \tag{46}$$

which depends upon a nonsingular $(\delta^T \tilde{\Delta} \delta)$ where δ may be the m or n-system. It is necessary to demonstrate that $(\delta^T \tilde{\Delta} \delta)$ is invertible if and only if the r-system is empty.

Firstly δ is assumed to be $\tilde{\Delta}$-orthogonal since if it was not initially, it could be transformed into this form by a change of basis (7), (8). When the r-system is empty then for any screw δ_i of the basis, $\delta_i^T \tilde{\Delta} \delta_i \neq 0$. Otherwise δ_i would also belong to the reciprocal system and the r-system would not be empty which is contrary to the hypothesis. Since δ is $\tilde{\Delta}$-orthogonal then $(\delta^T \tilde{\Delta} \delta)$ is diagonal and since none of the diagonal elements vanish the matrix is invertible. The only if case is demonstrated by merely reversing the order of the argument. Further, it follows that when the r-system is nonempty then $(\delta^T \tilde{\Delta} \delta)$ is singular and this method of filtering is not applicable.

However, this method is applicable when a cylindrical peg is constrained by a similarly shaped hole. Since the results have already been shown to satisfy the invariant conditions, there is no loss of generality in selecting the origin on the centerline which is also aligned with the z direction of the coordinate frame. The constraint and freedom of the body are specified by

$$a = \begin{bmatrix} \underline{i} & \underline{j} & \cdot & \cdot \\ \cdot & \cdot & \underline{i} & \underline{j} \end{bmatrix}, \quad B = \begin{bmatrix} \underline{k} & \cdot \\ \cdot & \underline{k} \end{bmatrix} \tag{47}$$

and a direct calculation of

$$P_a = a(a^T \tilde{\Delta} a)^{-1} a^T \tilde{\Delta} \tag{48}$$

$$p_B = B(B^T \tilde{\Delta} B)^{-1} B^T \tilde{\Delta} \tag{49}$$

yields

$$P_a = \begin{bmatrix} \underline{i} & \underline{j} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \underline{i} & \underline{j} & \cdot \end{bmatrix}, \quad p_B = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \underline{k} \\ \cdot & \cdot & \underline{k} & \cdot & \cdot & \cdot \end{bmatrix}. \tag{50}$$

The projector P_a filters out from the specified reaction wrench w any possible force or moment in the z direction. The projector p_B filters out from the specified twist T any possible translational or angular velocity in the x and y directions. It should be noted that "orthogonal" projection yields the same results for w^* and T^* when the origin is on the centerline. However, it is important to recognize that when the origin is displaced from the centerline the "orthogonal" projection solution will change whereas the one given here is invariant.

When both projectors in (50) are expressed in ray or axis coordinates then

$$P_a + P_b = I_6, \quad p_A + p_B = I_6 \tag{51}$$

which agrees with (30), (31). It is also useful to illustrate $\tilde{\Delta}$-orthogonal bases for the m and n-systems

$$\bar{a} = \begin{bmatrix} \underline{i} & \underline{i} & \underline{j} & \underline{j} \\ \underline{i} & -\underline{i} & \underline{j} & -\underline{j} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \underline{k} & -\underline{k} \\ \underline{k} & \underline{k} \end{bmatrix} \tag{52}$$

which are composed of screws with pitches ±1.

The kinestatic filtering based on $\tilde{\Delta}$ also may be derived in terms of a stationary value (SV) problem. Using (32), the two corresponding formulations are

$$\text{SV} : \quad \tfrac{1}{2} w^{\#T} \tilde{\Delta} w^{\#} \tag{53}$$

$$\text{SV} : \quad \tfrac{1}{2} T^{\#T} \tilde{\Delta} T^{\#} \tag{54}$$

where $w^{\#}$ and $T^{\#}$ are respectively the error wrench and error twist. The solutions are given by ε^{*}, λ^{*} in (22), (23) and w^{*}, T^{*} in (26), (27). It should be noted that since $\tilde{\Delta}$ is indefinite then the solutions to (53), (54) are not necessarily extremums.

The formulations (53), (54) afford an alternative viewpoint. Equation (53) represents one-half the mutual moment of the error wrench $w^{\#}$ with itself and (54) represents one-half the mutual moment of the error twist $T^{\#}$ with itself. Alternatively, using the pitch expressions in Chapter 3, (53) and (54) may be expressed as

$$\text{SV} : \quad \| w^{\#} \|^{2} \, h^{\#}_{w} \tag{55}$$

$$\text{SV} : \quad \| T^{\#} \|^{2} \, h^{\#}_{T} \tag{56}$$

where $h^{\#}_{w}$ and $h^{\#}_{T}$ are the pitches of the error wrench and error twist and $\| w^{\#} \|$, $\| T^{\#} \|$ are the respective magnitudes as in (5.2.78).

In addition to the invariant filtering based on $\tilde{\Delta}$, a second invariant kinestatic filter may be developed using the Euclidean Absolute presented in Section 3.3. Using a minimization formulation it may be expressed by

$$\text{minimize:} \quad w^{\#T} \tilde{\Gamma}_0 w^{\#} \tag{57}$$

$$\text{minimize:} \quad T^{\#} \tilde{\gamma}_0 T^{\#} \tag{58}$$

or equivalently

$$\text{minimize:} \quad (w - a\varepsilon^*)^T \tilde{\Gamma}_0 (w - a\varepsilon^*) \tag{59}$$

$$\text{minimize:} \quad (T - B\lambda^*)^T \tilde{\gamma}_0 (T - B\lambda^*) \tag{60}$$

the solutions to which are

$$\varepsilon^* = a^+ w = (a^T \tilde{\Gamma}_0 a)^{-1} a^T \tilde{\Gamma}_0 \tag{61}$$

$$\lambda^* = B^+ T = (B^T \tilde{\gamma}_0 B)^{-1} B^T \tilde{\gamma}_0 \tag{62}$$

and

$$w^* = a a^+ w = P_a w \tag{63}$$

$$T^* = B B^+ T = p_B T \tag{64}$$

where

$$\tilde{\Gamma}_0 = \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix}, \quad \tilde{\gamma}_0 = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix}. \tag{65}$$

are positive semidefinite matrices. Equations (57) and (58) may also be viewed as minimizing $\| w^{\#} \|^2$ and $\| T^{\#} \|^2$ respectively.

The solution of this formulation satisfies the three invariant conditions. Since it is based on the Euclidean Absolute, then any Euclidean collineation satisfies

$$\hat{k}^T \tilde{\Gamma}_0 \hat{k} = \tilde{\Gamma}_0 , \quad \hat{K}^T \tilde{\gamma}_0 \hat{K} = \tilde{\gamma}_0 \tag{66}$$

which may be used with (61) - (64) to demonstrate that the first condition is satisfied. It also may be readily verified that the change of scale (37) satisfies (66) to a scalar multiple and thus the second condition of invariance is satisifed. Further, from (38), (39) it is clear that the solution is also invariant with respect to a change in basis and thus the third condition of invariance is satisfied.

However, the solution is not very useful for direct application. This is because $\tilde{\Gamma}_0$ and $\tilde{\gamma}_0$ are each rank three, and for $m > 3$ or $n > 3$ respectively the matrices $(a^T \tilde{\Gamma}_0 a)$, $(B^T \tilde{\gamma}_0 B)$ are singular. Therefore from (61) - (64) a solution is not possible. For $m \leq 3$ or $n \leq 3$ the matrices will become singular respectively when an element in the corresponding basis is a line at infinity or when the screw axis directions are linearly dependent.

A third invariant kinestatic filter is obtained by combining the first two,

$$\tilde{\Psi} = \tfrac{1}{2}\tilde{\Delta} - \mu \tilde{\Gamma}_0 , \quad \tilde{\psi} = \tfrac{1}{2}\tilde{\Delta} - \mu \tilde{\gamma}_0 . \tag{67}$$

For simplicity, it is assumed that the arbitrary scalar μ has the same value in both expressions although in general this is not necessary. The corresponding stationary value formulations are

$$\mathrm{SV} : \quad w^{\#T} \tilde{\Psi} w^{\#} \tag{68}$$

$$\mathrm{SV} : \quad T^{\#T} \tilde{\psi} T^{\#} \tag{69}$$

the solutions of which are given by

$$\varepsilon^{*} = a^{+} w = (a^{T} \tilde{\Psi} a)^{-1} a^{T} \tilde{\Psi} \tag{70}$$

$$\lambda^{*} = B^{+} T = (B^{T} \tilde{\psi} B)^{-1} B^{T} \tilde{\psi} \tag{71}$$

and

$$w^{*} = a\, a^{+} w = P_{a} w \tag{72}$$

$$T^{*} = B\, B^{+} T = p_{B} T \tag{73}$$

When $\mu = 0$ and $\mu = \infty$ respectively this formulation reduces to the previous ones for $\tilde{\Delta}$ and $\tilde{\Gamma}_0$, $\tilde{\gamma}_0$. Further, because the formulation is a linear combination of the previous two, it is easily demonstrated that it satisfies the three invariant conditions.

It is also useful to express (68), (69) in the forms,

$$\mathrm{SV} : \quad \| w^{\#} \|^{2} (h_{w}^{\#} - \mu) \tag{74}$$

$$\mathrm{SV} : \quad \| T^{\#} \|^{2} (h_{T}^{\#} - \mu) \tag{75}$$

which are analogous to (55), (56) except that in each case the pitch of the error wrench or error twist is diminished by μ.

In order to determine the applicability of the formulation, it is necessary to calculate the eigenvalues for either $\tilde{\Psi}$ or $\tilde{\psi}$ which are equivalent since

$$\tilde{\Delta}\ \tilde{\Psi}\ \tilde{\Delta} = \tilde{\psi}. \tag{76}$$

The eigenvalue problem

$$|\Psi - \sigma\ I_6| = 0 \tag{77}$$

yields after some manipulation

$$(\sigma^2 + 2\mu\sigma - 1)^3 = 0. \tag{78}$$

The two distinct solutions are each of multiplicity three and

$$\sigma = \beta\ ,\quad -\beta^{-1} \tag{79}$$

where

$$\beta = \mu + (\mu^2 + 1)^{\frac{1}{2}}\ . \tag{80}$$

There are three positive and three negative eigenvalues and thus it is possible to have $\tilde{\Psi}$ and $\tilde{\psi}$-isotropic subspaces (of maximal rank three, see Porteous [1981] for example).

Analogous to the $\tilde{\Delta}$ formulation, when an isotropic subspace exists then the solution (70) - (73) is not applicable since the matrices

$$(a^T \tilde{\Psi}\ a)\ ,\quad (B^T \tilde{\psi}\ B) \tag{81}$$

are singular. Using the elements of B (or equivalently of a), then for a screw B_i to be $\tilde{\psi}$-isotropic it is necessary that

$$B_i^T \tilde{\psi}\ B_i = 0. \tag{82}$$

This equation may be satisfied in two distinct manners. The first case occurs when B_i is a screw of infinite pitch and then

$$B_i^T \tilde{\psi}\ B_i = B_i^T \tilde{\Delta}\ B_i + \mu\ B_i^T \tilde{\gamma}_0\ B_i = 0 + 0 \tag{83}$$

which is independent of μ. The second case occurs when B_i is a screw of finite pitch $h_i = \mu$ for which

$$B_i^T \tilde{\psi}\ B_i = \| B_i \|^2 (h_i - \mu) = 0. \tag{84}$$

From (83) it should be noted that if a screw of infinite pitch is in the isotropic subspace, then it will remain so under any variation of μ. However from (84), a screw of finite pitch in the isotropic subspace will no longer be a member when μ is varied. For example, when an isotropic subspace exists then,

$$|B^T \tilde{\psi}\ B| = |\tfrac{1}{2}B^T \tilde{\Delta}\ B - \mu\ B^T \tilde{\gamma}_0\ B| = 0. \tag{85}$$

Since $\tilde{\gamma}_0$ is of rank three, then the expansion of the $n \times n$ determinant yields a cubic in μ. For $n > 3$, the $n-3$ lead coefficients vanish which yields an n-3 multiplicity of $\mu = \infty$. In general, solving the cubic gives three possible corresponding isotropic subspaces.

Finally, it is believed that $\tilde{\Psi}$ and $\tilde{\psi}$ in (67) represent the most general matrices for which an unconstrained classical optimization formulation will yield invariant results. It is important to note that alternative types of invariant kinestatic filters are possible other than the ones presented here. However, a single invariant formulation which may be used in all instances has thus far not been determined. However, such a formulation may not even exist.

CHAPTER 6

SUMMARY AND CONCLUDING REMARKS

> I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery. R. Descartes [1637, p. 240]

A systematic presentation of geometric principles and methods has been developed from basic concepts. In Chapter 2, projective geometry was used as the starting point of the development. In order to work analytically from the outset, projective coordinates were introduced in a metric-free manner. This was then followed by a more general development of projective homogeneous coordinates using symmetrical determinant principles for points, planes and lines which were then extended to screws. This enabled a rather elegant description of dualistic properties. Projective transformations were introduced as those which left incidence properties invariant. The transformations for points, planes, lines and screws were united using an important tetrahedron principle based on incidence relations.

The systematic development continues in Chapter 3 for metrical geometry which is based directly on Cayley's principle of the Absolute. This allows a smooth and logical transition from projective geometry to metrical geometry via projective metrics. Following suggestions made by Clifford, determination of the pitches and axes of screws was generalized. This was a rediscovery of an earlier result given by Buchheim [1884a] who apparently was also directed by Clifford's work. Metrical collineations were introduced based upon the invariance of the Absolute. Further, definitions of norms and metrical coordinates enabled the introduction of new space elements with a magnitude. Elliptic geometry was detailed with the main result being to introduce the elliptic polarity. Projective and elliptic relations were identified for screws which appear similar in form. They are easily confused without thorough geometric understanding and an accompanying notation that delineates correlations from collineations and ray coordinates from axis coordinates.

Euclidean geometry was shown to have an asymmetry in its representation of duality because of the singular nature of the Euclidean Absolute. An elegant formulation for the pitch and axis of a screw was derived from Clifford's general principle. Most importantly, it appears that a new contribution has been made in the well-known subject of Euclidean geometry. That is, by a direct application of Cayley's Absolute, the group of Euclidean collineations is immediately deduced, namely translations, rotations and reflections through the origin.

Vector quantities were deduced based on one of Klein's principles. This allowed the introduction of new space elements which have both an associated magnitude and direction. In a manner, they enabled a resolution of the ambiguity of signs associated with metrical coordinates. The vector formulation of screws led to a form which is presently common. The role of twists and wrenches in mechanics was discussed along with the principle of virtual work.

Chapter 4 began with summary of the circumstances which motivated the work in this dissertation. Virtually all the material in this chapter is entirely original unless specifically noted otherwise. Important contributions include:

1. A basic mapping of screws onto the quadruple (h, r̲).
2. A quaternion mapping where the elliptic polarity induces a quaternion inverse. The operator role of the quaternion in relation to a screw and its polar is deduced.
3. A mapping of screws onto a four-dimensional inverse space where the elliptic polarity induces an inversion through the unit hypersphere followed by a reflection in a hyperplane.
4. A mapping of the quadruple (h, r̲) onto radial pencils and bundles of screws with constant pitch.
5. A three-way isomorphism between quaternions, points in an inversive space and radial pencils and bundles of screws.
6. A relation between pairs of polar helices and their torsion and curvature vectors.

Further, many of these developments were applied to Ball's planar representation of the two-system and contributions in this area include:

1. The representation was generalized to the inversive plane where the elliptic polarity induced an inversion through the unit circle followed by a reflection.

2. Points on the plane were characterized as radial pencils of screws and circles were characterized as pencils of cylindroids.
3. Specializations of quaternions represented ordinary complex numbers on the inversive plane.
4. A general derivation of the circle representation for the two-system was presented.
5. An examination of the five special two-systems in terms of their mappings.
6. Self-polar two-systems were deduced in terms of the planar mapping.
7. The apparently erroneous results from a complex number formulation were explained using the interpretation of the generalized planar mapping and exemplified its utility.
8. A special planar two-system was used to explain the effect of an origin translation on the elliptic polar of a screw.

Chapter 5 began with an explanation of hybrid control using the concept of kinestatic filtering. This was followed by an examination of a type of noninvariant hybird control and various types of invariant kinestatic filtering for application in hybrid control. Important contributions in this area include:

1. A demonstration that "orthogonal" projection is noninvariant with respect to Euclidean translations.
2. A demonstration that "orthogonal" projection is noninvariant with changes in unit length.
3. The determination of origin locations as a quadric surface where two screws are "orthogonal."
4. Three proposed necessary conditions for invariant kinestatic filtering.

5. The introduction of a family of invariant kinestatic filters using a variable parameter.
6. A detailed discussion on the applicability of the introduced invariant filters.

Further, the methods used for determining invariant kinestatic filters indicate that many more solutions may be obtained by optimizing invariant criteria.

It is the author's hope that the geometrical developments presented will provide a foundation for further utilization and popularization of screw theory in kinematics. Further, it is believed that the elucidation of the properties of the elliptic polarity and "orthogonal" projection will have a significant impact on the present noninvariant theory of hybrid control for robotic manipulators. Clearly, invariant methods of hybrid control are needed and this dissertation provides the geometrical foundations for their development as well as introducing initial invariant formulations which may be extended by optimization techniques.

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BIOGRAPHICAL SKETCH

The author's life history is an enigma.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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